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Generalized multiple integral representations for a large family of polynomials with applications

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ABSTRACT

This paper aims to provide a natural generalization and unification of a series of multiple integral representations for special classes of hypergeometric polynomials recently obtained by several authors. This generalization is obtained by considering a very large family of hypergeometric polynomials. The multiple integral representations given in this paper may be viewed as linearization relationship for the product of two different members of the associated family of hypergeometric polynomials.

1. Introduction

Almost four decades ago, Srivastava [1] introduced and investigated the following general family of polynomials:

$$S_n^N(z) := \sum_{k=0}^{[\frac{n}{N}]} \frac{(-n)_{Nk}}{k!} A_{n,k} z^k \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; N \in \mathbb{N}), \tag{1.1}$$

where $\{A_{n,k}\}_{n,k=0}^{\infty}$ is a suitably bounded double sequence of real or complex numbers, $[a]$ denotes the greatest integer $a \in \mathbb{R}$, and $(\lambda)_\nu$ denotes the Pochhammer symbol defined by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} \lambda(\lambda+1)\ldots(\lambda+n-1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}), \\ 1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}), \end{cases}$$

$\mathbb{N}$ and $\mathbb{C}$ being, as usual, the set of positive integers and the set of complex numbers, respectively. Moreover, it is understood conventionally that $(0)_0 := 1$.

This last family of polynomials and their variants as

$$S_{n,m}^N(z) := \sum_{k=0}^{[\frac{n}{N}]} \frac{(-n)_{Nk}}{k!} A_{m+n,k} z^k \quad (m, n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; N \in \mathbb{N}) \tag{1.2}$$

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were widely studied by Gonzalez et al. [2] and recently by Lin et al. [3] (see also Lin et al. [4,5]). It is easy to see that

$$S^{N}_{n,0}(z) = S^{N}_{n}(z).$$

Recently, Altin et al. [6] investigated the following family of bivariate polynomials:

$$S^{m,N}_{n}(x,y) := \sum_{k=0}^{[\frac{n}{N}]} A_{m+n,k} \frac{x^{n-Nk}}{(n-Nk)!} \frac{y^k}{k!} \qquad (m,n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; N \in \mathbb{N}), \tag{1.3}$$

which family contains as special cases the Lagrange–Hermite polynomials, the Lagrange polynomials and the Hermite–Kampé de Fériet polynomials. The reader should read [6–11] for further details.

It is easily observed by comparing definitions (1.1) and (1.2) that

$$S^{m,N}_{n}(x,y) = \frac{x^n}{n!} S^{N}_{n,m}\left(\frac{y}{(-x)^N}\right) \tag{1.4}$$

and

$$S^{N}_{n,m}(z)\Big|_{\frac{y}{(-x)^N}} = \frac{n!}{x^n} S^{m,N}_{n}(x,y). \tag{1.5}$$

This shows the fact that the two-variable polynomials $S^{m,N}_{n}(x,y)$ are essentially the same as the one-variable polynomials $S^{N}_{n,m}(z)$.

In the present investigation, we consider the following general family of polynomials:

$$\Xi^{N;(K_r);(L_p)}_{n;(T_s);(M_q)}(z;(\lambda_r),(\omega_r),(\alpha_p);(\mu_s),(\phi_s),(\beta_q)) = \Xi^{N;K_1,\ldots,K_r;L_1,\ldots,L_p}_{n;T_1,\ldots,T_s;M_1,\ldots,M_q}\left(z \,\middle|\, \begin{matrix} \lambda_1,\ldots,\lambda_r;\omega_1,\ldots,\omega_r;\alpha_1,\ldots,\alpha_p \\ \mu_1,\ldots,\mu_s;\phi_1,\ldots,\phi_s;\beta_1,\ldots,\beta_q \end{matrix}\right) \tag{1.6}$$

$$:= \sum_{j=0}^{[\frac{n}{N}]} \frac{(-n)_{Nj}\,(\lambda_1+\omega_1 n)_{K_1 j}\ldots(\lambda_r+\omega_r n)_{K_r j}\,(\alpha_1)_{L_1 j}\ldots(\alpha_p)_{L_p j}}{(\mu_1+\phi_1 n)_{T_1 j}\ldots(\mu_s+\phi_s n)_{T_s j}\,(\beta_1)_{M_1 j}\ldots(\beta_q)_{M_q j}} \frac{z^j}{j!} \tag{1.7}$$

$$((K_r),(L_p),(T_s),(M_q) \in \mathbb{N} \cup \{0\}; N \in \mathbb{N}; (\lambda_r),(\omega_r),(\alpha_p);(\mu_s),(\phi_s),(\beta_q) \in \mathbb{C})$$

where (and throughout this paper) (K_r) stands for the array of r parameters $K_1,\ldots,K_r$, with similar interpretations for (L_p), (T_s), (M_q), (λ_r), (ω_r), (α_p); (μ_s), (ϕ_s) and (β_q).

This general family of polynomials contains, as special cases, several other families of polynomials. For example, the following relationships hold between the polynomials $\Xi^{N;(K_r);(L_p)}_{n;(T_s);(M_q)}(z;(\lambda_r),(\omega_r),(\alpha_p);(\mu_s),(\phi_s),(\beta_q))$ and some simpler class of polynomials studied in [3,4,12–14]:

$$\mathcal{R}^{N,L_1,L_2}_{n,M_1,M_2}(z;\lambda_1,\lambda_2,\alpha_1,\alpha_2) := \sum_{j=0}^{[\frac{n}{N}]} \frac{(-n)_{Nj}\,(\lambda_1+n)_{L_1 j}\,(\lambda_2+n)_{L_2 j}}{(\alpha_1+1)_{M_1 j}\,(\alpha_2+1)_{M_2 j}} \frac{z^j}{j!} = \Xi^{N;L_1,L_2;—}_{n;M_1,M_2;—}\left(z \,\middle|\, \begin{matrix} \lambda_1,\lambda_2;1,0;— \\ \alpha_1+1,\alpha_2+1;0;— \end{matrix}\right), \tag{1.8}$$

$$\mathcal{S}^{L,M}_{n,N}(z;\lambda,\alpha) := \sum_{j=0}^{[\frac{n}{N}]} \frac{(-n)_{Nj}\,(\lambda+n)_{Lj}}{(\alpha+1)_{Mj}} \frac{z^j}{j!} = \Xi^{N;L;—}_{n;M;—}\left(z \,\middle|\, \begin{matrix} \lambda;1;— \\ \alpha+1;0;— \end{matrix}\right), \tag{1.9}$$

$$\mathcal{B}^{N,L}_{n,M}(z;\alpha,\beta,\gamma,\omega) := \sum_{j=0}^{[\frac{n}{N}]} \frac{(-n)_{Nj}\,(\gamma+\omega n)_{Lj}}{(\alpha+\beta n)_{Mj}} \frac{z^j}{j!} = \Xi^{N;L;—}_{n;M;—}\left(z \,\middle|\, \begin{matrix} \gamma;\omega;— \\ \alpha;\beta;— \end{matrix}\right), \tag{1.10}$$

$$\begin{aligned}\mathcal{T}^{N,L_1,L_2,L_3}_{n,M_1,M_2,M_3}(z;\lambda_1,\lambda_2,\lambda_3,\alpha_1,\alpha_2,\alpha_3) &:= \textstyle\sum_{j=0}^{[\frac{n}{N}]} \frac{(-n)_{Nj}\,(\lambda_1+n)_{L_1 j}\,(\lambda_2-n)_{L_2 j}\,(\lambda_3)_{L_3 j}}{(\alpha_1+n)_{M_1 j}\,(\alpha_2-n)_{M_2 j}\,(\alpha_3)_{M_3 j}} \frac{z^j}{j!} \\ &= \Xi^{N;L_1,L_2,L_3;—}_{n;M_1,M_2,M_3;—}\left(z \,\middle|\, \begin{matrix} \lambda_1,\lambda_2,\lambda_3;1,-1,0;— \\ \alpha_1,\alpha_2,\alpha_3;1,-1,0;— \end{matrix}\right),\end{aligned} \tag{1.11}$$

$$\begin{aligned}\Omega^{N;(K_r);(L_p)}_{n;(T_s);(M_q)}\left(z \,\middle|\, \begin{matrix} \lambda_1,\ldots,\lambda_r;\alpha_1,\ldots,\alpha_p \\ \mu_1,\ldots,\mu_s;\beta_1,\ldots,\beta_q \end{matrix}\right) &:= \textstyle\sum_{j=0}^{[\frac{n}{N}]} \frac{(-n)_{Nk}\,(\lambda_1+n)_{K_1 j}\ldots(\lambda_r+n)_{K_r j}\,(\alpha_1)_{L_1 j}\ldots(\alpha_p)_{L_p j}}{(\mu_1-n)_{T_1 j}\ldots(\mu_s-n)_{T_s j}\,(\beta_1)_{M_1 j}\ldots(\beta_q)_{M_q j}} \frac{z^j}{j!} \\ &= \Xi^{N;K_1,\ldots,K_r;L_1,\ldots,L_p}_{n;T_1,\ldots,T_s;M_1,\ldots,M_q}\left(z \,\middle|\, \begin{matrix} \lambda_1,\ldots,\lambda_r;1,\ldots,1;\alpha_1,\ldots,\alpha_p \\ \mu_1,\ldots,\mu_s;-1,\ldots,-1;\beta_1,\ldots,\beta_q \end{matrix}\right).\end{aligned} \tag{1.12}$$

The main object of this paper is to obtain a multiple integral representations associated with the polynomials defined by (1.6). These multiple integral representations generalize and unify the numerous results given recently by several authors, see for example [3,4,12–19]. Many special cases involving well-known families of polynomials are also given. As mentioned by Srivastava et al. [14], each integral representations derived in this paper may be viewed as a linearization relationship for the product of two different members of the associated family of hypergeometric polynomials.