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On semi-convergence of a class of relaxation methods for singular saddle point problems

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ABSTRACT

Recently, a class of efficient relaxation iterative methods has been proposed to solve the nonsingular saddle point problems. In this paper, we further prove the semi-convergence of these methods when it is applied to solve the singular saddle point problems. The semi-convergence properties of the relaxation iteration methods are carefully analyzed, which show that the iterative sequence generated by the relaxation iterative methods converges to a solution of the singular saddle point problem under suitable restrictions on the involved iteration parameters. In addition, numerical experiments are used to show the feasibility and effectiveness of the relaxation iterative methods for solving singular saddle point problems.

1. Introduction

In this paper, we consider a class of relaxation iterative methods for solving the following saddle point problem

$$\mathscr{A}\mathscr{X} = \begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ -g \end{pmatrix} = b, \tag{1.1}$$

where $A \in \mathbb{R}^{m\times m}$ is a symmetric positive definite matrix, and $B \in \mathbb{R}^{m\times n}$ is a rectangular matrix satisfying $m \geq n$, and $b \in \mathbb{R}$ is a given vector in the range of matrix $\mathscr{A} \in \mathbb{R}^{m+n}$, with $f \in \mathbb{R}^m$ and $g \in \mathbb{R}^n$. This kind of system of linear equations arises in a variety of scientific and engineering applications, such as computational fluid dynamics, constrained optimization, optimal control, weighted least-squares problems, electronic networks, computer graphic, the constrained least squares problems and generalized least squares problems etc; see [2,9,18,23,28,29] and the references therein. In addition, we can also obtain saddle point linear systems from the meshfree discretization of some partial differential equations [12,20] or the mixed hybrid finite element discretization of second order elliptic problems [11]. A comprehensive summary about various applications leading to saddle point matrices and a general framework of preconditioning methods and their theoretical analyses were given in [8].

When B is of full column rank, the linear system (1.1) is nonsingular. In this case, large varieties of effective iterative methods based on matrix splitting and their numerical properties have been proposed and discussed: Golub et al. [19] proposed the successive overrelaxation-like method, Bai et al. [2] proposed the generalized successive overrelaxation method and the generalized inexact accelerated overrelaxation method, Bai and Wang [3] proposed the parameterized inexact Uzawa method, Li et al. [21] proposed the accelerated overrelaxation-like method, Shao et al. [24] proposed the modified SOR-like method,

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Zheng et al. [34] and Darvishi and Hessari [16] proposed and studied the symmetric successive overrelaxation-like method, Wu et al. [25] proposed the modified SSOR-like method, Chao et al. [15] studied the generalized symmetric successive overrelaxation method, Yun [30] proposed the Uzawa symmetric accelerated overrelaxation method and so forth. See, for example [1,4–7].

When matrix B is column rank-deficient, i.e., $rank(B) < n \le m$, the matrix $\mathscr{A}$ is singular. We call (1.1) the singular saddle point problem. A number of effective methods have been proposed in the literature to solve the singular saddle point problems, such as the Uzawa-type methods [22,32,35], Krylov subspace methods [26,33] and matrix splitting iteration methods [13,14,17].

Owing to the high efficiency of Uzawa-type method for solving the saddle point problem, Yun [31] recently proposed a modified inexact accelerated overrelaxation-like iterative method to solve nonsingular saddle point problems. The iteration scheme of MIAOR method is of the form

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = H(\alpha, r, \omega)\begin{pmatrix} x_k \\ y_k \end{pmatrix} + M(\alpha, r, \omega)\begin{pmatrix} f \\ -g \end{pmatrix} \tag{1.2}$$

where

$$H(\alpha, r, \omega) = \begin{pmatrix} I - \omega P^{-1}A & -\omega P^{-1}B \\ \frac{\omega}{1-\alpha r} Q^{-1}B^T(I - rP^{-1}A) & \frac{-r\omega}{1-\alpha r} Q^{-1}B^T P^{-1}B + I \end{pmatrix},$$

and

$$M(\alpha, r, \omega) = \omega\begin{pmatrix} P^{-1} & 0 \\ \frac{r}{1-\alpha r} Q^{-1}B^T P^{-1} & \frac{1}{1-\alpha r} Q^{-1} \end{pmatrix},$$

with $\omega > 0$ and $(1 - \alpha r) \neq 0$ and $P \in \mathbb{R}^{m\times m}$ being a symmetric positive definite matrix which approximates A, $Q \in \mathbb{R}^{n\times n}$ being a symmetric positive definite matrix which approximates the Schur complement matrix $B^T P^{-1} B$, and $\alpha + \beta = 1$ with $0 \le \alpha \le 1$. For singular saddle point problems, the Schur complement $B^T P^{-1} B$ is singular. Hence, we select Q in (1.2) as a symmetric positive semi-definite (and a potentially singular) matrix. Then, the MIAOR method can be generalized to the following form:

Method 1.1 (The MIAOR iteration method)**.** Given initial guesses $x_0 \in \mathbb{R}^m$ and $y_0 \in \mathbb{R}^n$, for $k = 0, 1, 2, \ldots,$ until $x_k \in \mathbb{R}^m$ and $y_k \in \mathbb{R}^n$ converge

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = H(\alpha, r, \omega)\begin{pmatrix} x_k \\ y_k \end{pmatrix} + M(\alpha, r, \omega)\begin{pmatrix} f \\ -g \end{pmatrix} \tag{1.3}$$

where

$$H(\alpha, r, \omega) = \begin{pmatrix} I - \omega P^{-1}A & -\omega P^{-1}B \\ \frac{\omega}{1-\alpha r} Q^{\dagger}B^T(I - rP^{-1}A) & \frac{-r\omega}{1-\alpha r} Q^{\dagger}B^T P^{-1}B + I \end{pmatrix},$$

and

$$M(\alpha, r, \omega) = \omega\begin{pmatrix} P^{-1} & 0 \\ \frac{r}{1-\alpha r} Q^{\dagger}B^T P^{-1} & \frac{1}{1-\alpha r} Q^{\dagger} \end{pmatrix},$$

with $\omega > 0$ and $(1 - \alpha r) \neq 0$ and $P \in \mathbb{R}^{m\times m}$ as a symmetric positive definite matrix which approximates A, $Q \in \mathbb{R}^{n\times n}$ as a symmetric positive semi-definite (and a potentially singular) matrix which approximates the Schur complement matrix $B^T P^{-1} B$, $Q^{\dagger}$ denotes the Moore–Penrose pseudoinverse of matrix Q, and $\alpha + \beta = 1$ with $0 \le \alpha \le 1$.

Since the MIAOR method is quite promising compared with the MAOR, GSSOR and USSOR as well as the GSOR methods in [31], in this paper we employ the MIAOR iteration method to solve singular saddle point problems. Throughout this paper, x^* denotes the conjugate transpose of the vector x. $\lambda_{\min}(H)$ and $\lambda_{\max}(H)$ denote the minimum and maximum eigenvalues of the Hermitian matrix H, respectively. For a square matrix G, N(G) denotes the null space of G; $\sigma(G)$ denotes the set of all eigenvalues of G, and $\rho(G)$ denotes the spectral radius of G.

The remainder of this paper is organized as follows. In Section 2, we introduce several useful lemmas. Based on these lemmas, the semi-convergence properties of the MIAOR method used to solve singular saddle point problems are analyzed in Section 3. Then the semi-convergence of GSSOR and USSOR methods are further discussed in Section 4. In Section 5, we test the robustness and the efficiency of the MIAOR iteration method by comparing its iteration number, elapsed CPU time and relative errors with those of the GSOR [2] and MAOR methods. Finally in Section 6, we give a brief conclusion of the paper to end this work.

2. Preliminary knowledge

To begin, we introduce some preliminary knowledge which will be used in the analysis of the semi-convergence properties of the MIAOR iteration method.

Let $B = U(B_r, 0)V^*$ be the SVD decomposition of B, where $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are two unitary matrices, $B_r = (\Sigma_r, 0)^* \in \mathbb{C}^{m\times r}$ where r denotes the rank of matrix B and $\Sigma_r = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r)$ where σ_i is a singular value of matrix B. Let $U = (U_1, U_2)$ where $U_1 \in \mathbb{C}^{m\times r}$ and $U_2 \in \mathbb{C}^{m\times(m-r)}$, and the matrix V be partitioned into (V_1, V_2), correspondingly. Then