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Parametrized means and limit properties of their Gaussian iterations

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ABSTRACT

We propose the notion of a mean depending on a parameter, define Gauss-type iterates of parametrized mean-type mapping and study their limit behaviour. Also a suitable invariance property of the limit function is established. In such a way we obtain an extension of generalized Gaussian algorithm (Theorem 2.5) to the case of a compact set of parameters. In particular, we generalize some well known results of J. Matkowski dealing with iterates of mean-type mappings not depending on parameter.

1. Introduction

Given an interval I of reals and a positive integer p a function $M: I^p \to I$ is called a *mean on* I^p if

$$\min\{x_1, \dots, x_p\} \le M(x_1, \dots, x_p) \le \max\{x_1, \dots, x_p\}$$

for every point $(x_1, \dots, x_p) := I^p$; see, for instance, [4] by Bullen. The mean M is said to be *strict* if the above inequalities are sharp whenever $\min\{x_1, \dots, x_p\} < \max\{x_1, \dots, x_p\}$. The classical examples, well known already in the antiquity and basically important also for applications, are the arithmetic mean $A : \mathbb{R}^p \to \mathbb{R}$:

$$A(x_1, \dots, x_p) = \frac{x_1 + \dots + x_p}{p},$$

the geometric mean G: $[0, \infty)^p \to [0, \infty)$:

$$G(x_1, \dots, x_p) = \sqrt[p]{x_1 \dots x_p},$$

and the harmonic mean H: $(0, \infty)^p \to (0, \infty)$:

$$H(x_1, \dots, x_p) = \frac{p}{\frac{1}{x_1} + \dots + \frac{1}{x_p}}.$$

A wealth of information on means was presented by Hardy, et al. [16], by Borwein and Borwein [3], and by Daróczy and Páles [11]. Means are mostly assumed there to be continuous functions which, in general, is not the case in the present paper.

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Mean values are still alive and fresh, also if we have in mind their application to some other fields and topics. Theory of summability serves as a good example. There, for instance, weighted de la Vallée Poussin means provide a useful tool in proving some new generalized Tauberian-like theorems. Also the weighted mean summability method has turned out to be helpful in improving so classical field as theory of summability (see, among others, [5,6] by Çanak and [7,8,27] by Çanak and Totur).

It was Lagrange [17], and then mainly also Gauss (see [15]), who observed a close connection of elliptic integrals to mean iterations and so-called invariant means. Taking any $x, y := (0, \infty)$ and putting

$$x_1 = x, \quad y_1 = y$$

and

$$x_{n+1} = A(x_n, y_n), \quad y_{n+1} = G(x_n, y_n) \tag{1.1}$$

for every $n \in \mathbb{N}$, they proved that both the sequences converge to a common limit, say $A \otimes G(x, y)$. The function $A \otimes G$ is a mean on $(0, \infty)$, named by them the *arithmetic–geometric mean* (*medium arithmeticum–geometricum*). One of its crucial property is the (A, G)-invariance:

$$A \otimes G\left(\frac{x+y}{2}, \sqrt{xy}\right) = A \otimes G(x, y), \quad x, y \in (0, \infty).$$

Another one is surprising equality

$$A \otimes G(x, y) = \left(\frac{2}{\pi}\int_0^{\pi/2} \frac{1}{\sqrt{x^2\cos^2 t + y^2\sin^2 t}}\mathrm{d}t\right)^{-1} \tag{1.2}$$

held for all $x, y := (0, \infty)$, discovered by Gauss (cf. [15], [3] and [11]; also [20] by Matkowski) but probably known already for Lagrange. It is known that Gaussian iterations (1.1) typically converge quadratically (see, for instance, [3]), so the rate of their convergence is pretty high. Thus, according to (1.2), they allow to get a good approximation of the integral in that formula. This serves as an example of possible applications of Gaussian iterations (1.2) in computational mathematics and numerical analysis (cf. [3]). For further examples of an important role of the arithmetic-geometric mean in the Gauss theory of elliptic integrals and functions the reader is referred, among others, to the book [3], and papers [10] by Cox and [1] due to Almkvist and Berndt. Some geometric aspects of the Gauss algorithm have been described by Schoenberg in [24].

Observe that iterating the map (A, G): $(0, \infty)^2 \to (0, \infty)^2$ one can rewrite the convergence of Gaussian iterates to the mean $A \otimes G$ in the brief form

$$(A, G)^n \to (A \otimes G, A \otimes G).$$

The Gauss procedure was fairly extended. It seems that for the first time the theorem below was formulated by J. Matkowski. In 1999 he proved it for $p = 2$, under slightly stronger assumptions of the strictness of at least one of the involved means (see [20]). Its general version was established by him 10 years later in [22].

Here and in what follows we denote by Δ_p the diagonal $\{\boldsymbol{x} \in I^p : x_1 = \cdots = x_p\}$.

Generalized Gaussian algorithm. Let I be an interval of reals and let $M_1, \ldots, M_p : I^p \to I$ be continuous means such that

$$\min\{M_1(\boldsymbol{x}), \ldots, M_p(\boldsymbol{x})\} = \min\{x_1, \ldots, x_p\}$$

and

$$\max\{M_1(\boldsymbol{x}), \ldots, M_p(\boldsymbol{x})\} = \max\{x_1, \ldots, x_p\}$$

imply $\boldsymbol{x} := \Delta_p$ for every $\boldsymbol{x} = (x_1, \ldots, x_p) := I^p$. Then there is a continuous mean K: $I^p \to I$ such that

$$\lim_{n\to\infty}(M_1, \ldots, M_p)^n \to (K, \ldots, K)$$

uniformly on every compact subset of I^p; moreover, K is the unique continuous $(M_1, \ldots, M_p)$-invariant mean:

$$K \circ (M_1, \ldots, M_p) = K.$$

Many information about the Gaussian algorithm as well as a version of the above theorem can be found in the expository article [11] by Daróczy and Páles, published in 2002. Recently, J. Matkowski generalized the "moreover" part of the theorem proving the uniqueness of the $(M_1, \ldots, M_p)$-invariant mean in the class of all, not necessarily continuous, means on I^p (see [23]). Observe that the assumption imposed on the means $M_1, \ldots, M_p$ in generalized Gaussian algorithm is satisfied whenever at most one of them is not strict.

Nevertheless the history of the generalization of the Gauss procedure is pretty longer. Still in 1987 in the book [3] Borwein and Borwein, extending some ideas of Lehmer [19], Schoenberg [24], and Foster and Phillips [14], proved the following generalization of the Gauss procedure. Unfortunately, to realize it one should collect some partial results, viz. [3, Theorems 8.2 and 8.3 together with two sentences just before Theorem 8.2, Example 1 on p. 247, Theorem 88, the paragraph just before Comments and Exercises on p. 269 and Example 7 on p. 272].