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Defect corrected averaging for highly oscillatory problems

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ABSTRACT

The accurate solution of partial differential equations with highly oscillatory source terms over long time scales constitutes a challenging problem. There exists a variety of methods dealing with problems where there are processes, equations or variables on fine and coarse scales. Multiscale methods have in common, that they neither fully resolve the fine scale, nor completely ignore it. On the one hand, these methods strive, without significantly sacrificing accuracy or essential properties of the system, to be much more efficient than methods that fully resolve the fine scale. On the other hand, these methods should be considerably more accurate than methods that completely ignore the fine scale. Our defect corrected averaging procedure is based on a modified coarse scale problem, that approximates the solution of the fine scale problem in stroboscopic points. Nevertheless, our approximation process is clearly different from the stroboscopic averaging method. We give an error estimate for the solution of the modified problem. The computational efficiency of the approximation is furthermore improved by the application of preconditioning techniques. Tests on numerical examples show the efficiency and reliability of our approach.

1. Introduction

Partial differential equations (PDEs) with a highly oscillatory temporal source term constitute challenging problems for numerical time integrators. A variety of methods has been developed in order to treat these problems without resolving the microscale oscillations, among them the heterogeneous multiscale framework [1] as well as several averaging concepts like stroboscopic averaging [2], and the mollified impulse method [3].

In the methodology of the heterogeneous multiscale framework there is a microscopic model and a macroscopic model of the problem. Compression and reconstruction operators serve to switch between these two state spaces. The aim is to accurately approximate the macroscopic state of the system, see [1]. Thus, a macroscopic scheme is used to approximate the macroscopic state, whereas the effects on the microscopic scale have to be taken into account in order to update the macroscopic state. Our defect corrected averaging approach can be seen as a special case of this methodology, although we feel that it is more accurately summarized under the concept of stroboscopic averaging.

Within the framework of stroboscopic averaging, the flow of the original highly oscillatory problem is approximated by the flow of an averaged problem, where the accuracy is high at so-called stroboscopic points. Each function evaluation during the

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integration of the averaged macroscale problem is approximated by microscale simulations of the underlying highly oscillatory problem over some periods and an appropriate averaging. In the case of semidiscretized PDEs, the computational cost of the stroboscopic averaging approach increases strongly with the size of the semidiscretized system, rendering the method less useful.

In this paper, we consider initial value problems (IVPs) for systems of linear ODEs containing a highly oscillatory periodic source term $s(t)$ with small period size $\varepsilon > 0$

$$\begin{aligned} y'(t) &= Ly(t) + s(t), \quad y(t_0) = y_0 \in \mathbb{R}^m, \\ s(t) &= s(t+\epsilon). \end{aligned} \tag{1}$$

Such problems arise from the semidiscretization of parabolic PDEs with a highly oscillatory, periodic temporal source term such as considered in Section 3.

A simple way to deal with the fine scale term is to average the source term over an interval of length ε

$$\bar{s} = \frac{1}{\epsilon}\int_0^{\epsilon} s(t)dt,$$

and solve the averaged ODE

$$\bar{y}'(t) = L\bar{y} + \bar{s}, \quad \bar{y}(t_0) = y_0, \tag{2}$$

which is independent of the small scale parameter ε. Linear ODEs of this class can be solved with standard integrators, or with specialized integrators like exponential integrators [4]. In this paper we present and investigate, in Section 2, a *defect corrected averaging method* for problem (1). The aim of this method is to reduce the error $y(t) - \bar{y}(t)$ between the solutions of (1) and (2) in the stroboscopic time points $t = n\varepsilon$. The method is further improved by the application of preconditioning techniques. After that, a model problem is discussed in Section 3 and the accuracy and computational efficiency of the new method with respect to standard stiff integrators is shown in Section 4. Finally, we discuss our results and present conclusions in Section 5.

2. The defect corrected averaging method

2.1. Exact solution of linear ODEs with periodic source at stroboscopic points

The exact solution of the IVP for a linear ODE system with time-dependent source $f(t)$, that is

$$y'(t) = Ly(t) + f(t), \quad y(0) = y_0, \tag{3}$$

is given by

$$y(t) = \exp(tL)y_0 + \exp(tL)\int_0^t \exp(-\tau L)f(\tau)d\tau. \tag{4}$$

For a polynomial source term in (3), i.e. $f(t) = \sum_{k=0}^{K} g_k t^k$, there is, see [5], an explicit representation of the solution in the form

$$y(t) = \phi_0(tL)y_0 + t\sum_{k=0}^{K}\phi_{k+1}(tL)g_k t^k. \tag{5}$$

Here the analytic functions $\phi_k(z)$ are recursively defined by

$$\phi_0(z) = \exp(z), \quad \phi_1(z) = \frac{\phi_0(z) - 1}{z}, \tag{6}$$

$$\phi_{k+1}(z) = \frac{\phi_k(z) - 1/k!}{z}, \quad k \geq 1. \tag{7}$$

In the case of a constant source term $f(t) = g_0$, the solution of (3) is thus given by

$$y(t) = \phi_0(tL)y_0 + t\phi_1(tL)g_0. \tag{8}$$

The representation of the exact solution of (3) with polynomial source term $f(t)$ by functions ϕ_k, $k \geq 0$, is applied in the derivation of adaptive Runge–Kutta methods, where ϕ_0 is chosen as a rational approximation to the exponential, see [5], as well as in the derivation of exponential integrators, where ϕ_0 is assumed to be the exponential function itself, see [4].

Now we consider problem (3) with an ε-periodic source term f. We rewrite its solution (4) at the stroboscopic point $t = n\varepsilon$ in the form

$$y(t) = y(n\varepsilon) = \exp(tL)y_0 + \exp(tL)\sum_{k=0}^{n-1}\exp(-k\varepsilon L)\int_0^{\varepsilon}\exp(-\tau L)f(\tau)d\tau \tag{9}$$

$$= \exp(tL)y_0 + \frac{\exp(tL) - 1}{1 - \exp(-\varepsilon L)}\int_0^{\varepsilon}\exp(-\tau L)f(\tau)d\tau \tag{10}$$