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Elliptic extensions of the Apostol–Bernoulli and Apostol–Euler polynomials

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ABSTRACT

In this paper, we investigate the elliptic analogues of the Apostol–Bernoulli and Apostol–Euler polynomials and obtain the closed expressions of sums of products for these elliptic type polynomials. Some interesting special cases are also shown.

1. Introduction

The classical Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$ together with their familiar generalizations $B_n^{(\alpha)}(x)$ and $E_n^{(\alpha)}(x)$ of (real or complex) order α are usually defined by means of the following generating functions (see, for details, [21, pp. 25–32] and [26, pp. 59–66]):

$$\left(\frac{z}{e^z-1}\right)^\alpha e^{xz} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{z^n}{n!} \quad (|z|<2\pi) \tag{1.1}$$

and

$$\left(\frac{2}{e^z+1}\right)^\alpha e^{xz} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{z^n}{n!} \quad (|z|<\pi). \tag{1.2}$$

Obviously, the classical Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$ are respectively defined by

$$B_n(x) := B_n^{(1)}(x) \quad \text{and} \quad E_n(x) := E_n^{(1)}(x) \quad (n \in \mathbb{N}_0),$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{N} = \{1, 2, \ldots\}$.

The Bernoulli numbers B_n and Euler numbers E_n are respectively defined by

$$B_n := B_n(0) \quad \text{and} \quad E_n := 2^n E_n\left(\frac{1}{2}\right).$$

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Some interesting analogues of the classical Bernoulli polynomials and numbers were first investigated by Apostol [1, p. 165, Eq. (3.1)] and (more recently) by Srivastava [24, pp. 83–84]. Here we begin by recalling Apostol's definitions as follows:

Definition 1.1 (Apostol [1]; see also Srivastava [24])**.** The Apostol–Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$ in x are defined by means of the generating function:

$$\frac{ze^{xz}}{\lambda e^z-1}=\sum_{n=0}^{\infty}\mathcal{B}_n(x;\lambda)\frac{z^n}{n!} \tag{1.3}$$
$$(|z|<2\pi \text{ when } \lambda=1;\ |z|<|\log\lambda| \text{ when } \lambda\neq 1)$$

with, of course,

$$B_n(x)=\mathcal{B}_n(x;1) \quad \text{and} \quad \mathcal{B}_n(\lambda):=\mathcal{B}_n(0;\lambda),$$

where $\mathcal{B}_n(\lambda)$ denotes the so-called Apostol–Bernoulli numbers (in fact, it is a function in λ).

Recently, Luo and Srivastava extended further the so-called Apostol–Euler polynomials as follows:

Definition 1.2 (Luo and Srivastava [15] or Luo [16])**.** The Apostol–Euler polynomials $\mathcal{E}_n(x;\lambda)$ in x are defined by means of the generating function:

$$\frac{2e^{xz}}{\lambda e^z+1}=\sum_{n=0}^{\infty}\mathcal{E}_n(x;\lambda)\frac{z^n}{n!} \qquad (|z|<|\log(-\lambda)|), \tag{1.4}$$

with, of course,

$$E_n(x)=\mathcal{E}_n(x;1) \quad \text{and} \quad \mathcal{E}_n(\lambda):=2^n\mathcal{E}_n\left(\frac{1}{2};\lambda\right),$$

where $\mathcal{E}_n(\lambda)$ denote the so-called Apostol–Euler numbers (in fact, it is a function in λ).

The Apostol–Bernoulli polynomials and Apostol–Euler polynomials have been investigated by many authors (see, e.g., [1,2,4,6,8,13,15–19,24,25,27]). The formulas of sums of products for the Bernoulli and Euler polynomials and numbers have been investigated in the references [5,7,10,12,23]. Recently, Ivashkevich et al. and Machide introduced Kronecker's double series as an elliptic analogue for the Bernoulli polynomials and also gave their the formulas of sums of products (see, [11,20]).

In the present paper, we define the elliptic Apostol–Bernoulli polynomials and elliptic Apostol–Euler polynomials by means of the corresponding generalized Eisenstein summation and generalized Jacobi's theta function. We obtain the formulas of sums of products for the elliptic Apostol–Bernoulli and elliptic Apostol–Euler polynomials.

The paper is organized as follows: In the second section we define the elliptic Apostol–Bernoulli polynomials and obtain the formula of sums of products for the elliptic Apostol–Bernoulli polynomials. In the third section we define the elliptic Apostol–Euler polynomials and show some formulas for the elliptic Apostol–Euler polynomials.

2. Elliptic Apostol–Bernoulli polynomials

In this section we define the elliptic Apostol–Bernoulli polynomials and give the formulas for sums of products of the elliptic Apostol–Bernoulli polynomials. Some special cases are also shown.

We will use some standard notation: $H:=\{\tau\in\mathbb{C}|\tau:\Im\tau>0\}, e(t):=\exp(2\pi it), q=e(\tau), z=e(\xi), w=e(x)$. The classical Jacobi's theta functions [21, p. 371] are defined by

$$\vartheta_1(x;\tau)=\sum_{n\in\mathbb{Z}}e\left(\frac{1}{2}\left(n+\frac{1}{2}\right)^2\tau+\left(n+\frac{1}{2}\right)\left(x+\frac{1}{2}\right)\right), \tag{2.1}$$

$$\vartheta_2(x;\tau)=i\sum_{n\in\mathbb{Z}}(-1)^n e\left(\frac{1}{2}\left(n+\frac{1}{2}\right)^2\tau+\left(n+\frac{1}{2}\right)\left(x+\frac{1}{2}\right)\right), \tag{2.2}$$

where $\mathbb{Z}$ denotes the set of integers.

From (2.1) and (2.2), we easily verify the following quasi periodicity properties:

$$\vartheta_1(x+1;\tau)=-\vartheta_1(x;\tau), \qquad \vartheta_1(x+\tau;\tau)=-e\left(-x-\frac{\tau}{2}\right)\vartheta_1(x;\tau), \tag{2.3}$$

$$\vartheta_2(x+1;\tau)=-\vartheta_2(x;\tau), \qquad \vartheta_2(x+\tau;\tau)=e\left(-x-\frac{\tau}{2}\right)\vartheta_2(x;\tau). \tag{2.4}$$