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On the construction of integrable surfaces on Lie groups

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ABSTRACT

The problem of the immersion of a two-dimensional surface into a three-dimensional Euclidean space can be formulated in terms of the immersion of surfaces in Lie groups and Lie algebras. A general formalism for this problem is developed, as well as an equivalent Mauer–Cartan system of differential forms. The particular case of the Lie group $SU(2)$ is examined, and it is shown to be useful for studying integrable surfaces. Some examples of such surfaces and their equations are presented at the end, in particular, the cases of constant mean curvature and of zero Gaussian curvature.

1. Introduction

A domain $S \subset \mathbb{R}^2$ can be immersed into three-dimensional Euclidean space by means of a mapping $\mathbf{F} = (F_1, F_2, F_3) : \rho \to \mathbb{R}^3$. The natural Euclidean metric on $\mathbb{R}^3$ will then induce some metric g on this surface. In the case in which the surface is sufficiently sufficiently smooth, for $(u, v) \in S$, the functions $g_{ij}(u, v)$ which define the first fundamental form and the functions $h_{ij}(u, v)$ which define the second fundamental form satisfy a system of three nonlinear equations known as the Gauss–Codazzi equations. The Gauss–Weingarten equations can be thought of as providing a compatibility condition for generating these equations [1,2]. There exist two geometrical quantities which characterize such surfaces and these are the mean curvature H and the Gaussian curvature K. There is a great deal of interest mathematically in characterizing surfaces which possess a particular value of curvature H or K. The cases in which either of H or K is constant have been studied in their own right [3–6]. It has been established that the fundamental forms of such surfaces can be expressed in terms of solutions of the sine-Gordon and sinh-Gordon equations. These two equations are in fact known to be integrable and as well, large classes of solutions can be obtained for them explicitly. These solutions in turn permit the different global properties of their associated surfaces to be studied, such as the question of Hopf [7]: is a compact surface in $\mathbb{R}^3$ with constant mean curvature necessarily a sphere.

The immersion of a two-dimensional surface into a three-dimensional Euclidean space, as well as the n-dimensional generalization of this problem can be related to the study of surfaces in Lie groups and Lie algebras [8–11]. A surface is called integrable if and only if its Gauss–Codazzi equations are integrable [12]. Integrable equations also result from the compatibility condition of a pair of linear equations, and this pair is generally referred to as a Lax pair. A novel feature of the present work is that a formulation of surfaces on Lie groups and Lie algebras can be approached by means of Cartan's method of moving frames. An outline of the method will be presented and it will be shown that equations which result from the Gauss–Codazzi, Lax pair approach can also be obtained by defining a suitable set of differential forms. These two approaches will be compared and although some of the results have appeared before [10], proofs for all of the theorems introduced here will be given in detail. It is consequently shown that the problem of the immersion of a two-dimensional surface into a three-dimensional Euclidean space is related to the problem

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of studying surfaces in Lie groups and in Lie algebras and can be based on either of these two formalisms. The use of Lie point groups in fact provides an effective method for finding as well as classifying integrable surfaces [13]. Finally, the results that are obtained by way of these approaches will be applied to the study of some integrable surfaces. Starting for example from a suitable Lax pair, it is possible to construct explicitly large classes of integrable surfaces. To this end, the case of surfaces with constant mean curvature is considered. Finally, surfaces with zero Gaussian curvature are shown to be associated with the Born–Infeld equation [14].

2. Determination of the first and second fundamental forms

First it will be shown that the immersion of a two-dimensional generalization of this problem can actually be regarded as a study of surfaces in Lie groups and Lie algebras. It will be shown subsequently that identical results can be obtained from an approach based on differential forms.

Suppose $\mathcal{G}$ is the Lie algebra of a Lie group G in which there exists an invariant scalar product which is not necessarily positive definite. Such a scalar product exists for any semi-simple, finite dimensional group as well as for many infinite dimensional groups. It will be denoted as $\langle g_1, g_2 \rangle$ for $g_1, g_2 \in \mathcal{G}$. Let $\{e_i\}$ be an orthonormal basis in $\mathcal{G}$ so $\langle e_i, e_j \rangle = \delta_{ij}$. This will exist when the scalar product is not degenerate and $\dim \mathcal{G} = n < \infty$.

Introduce the function $\Phi(u, v) \in G$ for every (u, v) in some neighborhood of $S \subset \mathbb{R}^2$. There exists a canonical map from the tangent space of G to the Lie algebra $\mathcal{G}$. Given the pair of tangent vectors $\partial\Phi/\partial u$ and $\partial\Phi/\partial v$ of Φ at (u, v), the mapping Φ can be defined by means of the pair

$$\frac{\partial \Phi}{\partial u} = U\Phi, \quad \frac{\partial \Phi}{\partial v} = V\Phi, \tag{2.1}$$

where,

$$U = U_j e_j, \quad V = V_j e_j. \tag{2.2}$$

In (2.2), U_j and V_j are functions of (u, v) and the summation convention is used throughout. Therefore, (2.1) defines Φ by its values in the Lie algebra $\mathcal{G}$.

Theorem 2.1. *Let $\Phi(u, v) \in G$ be a differentiable function of u and v for every $(u, v) \in S \subset \mathbb{R}^2$, S a neighborhood of $\mathbb{R}^2$. Then Φ which is defined by (2.1) and (2.2) exists if and only if the functionsU_j and V_j satisfy*

$$\frac{\partial U_j}{\partial v} - \frac{\partial V_j}{\partial u} = V_k U_l c_{klj}. \tag{2.3}$$

The c_{klj} in (2.3) are the structural constants of $\mathcal{G}$ which are given by

$$[e_k, e_l] = c_{klj} e_j, \quad k, l, j = 1, \cdots, n. \tag{2.4}$$

Proof. Differentiate the first equation in (2.1) with respect to v and the second equation with respect to u. Upon subtracting these derivatives, it is found that the following expression vanishes,

$$\frac{\partial U_j}{\partial v} e_j \Phi + U_j e_j \frac{\partial \Phi}{\partial v} - \frac{\partial V_j}{\partial u} e_j \Phi - V_j e_j \frac{\partial \Phi}{\partial u} = \left(\frac{\partial U_j}{\partial v} - \frac{\partial V_j}{\partial u} \right) e_j \Phi + U_l V_s (e_l e_s - e_s e_l) \Phi = 0.$$

Upon substitution of the bracket (2.4) into this, the coefficient of e_j is exactly Eq. (2.3). □

The mapping Φ acts to induce a surface in G, and a surface can also be introduced in the Lie algebra as well. Let $\mathbf{F}(u, v) \in \mathcal{G}$ for every (u, v) in a neighborhood S of $\mathbb{R}^2$. The surface will be characterized by the first and second fundamental forms. The first fundamental form of $\mathbf{F}$ is defined to be

$$I = \left\langle \frac{\partial \mathbf{F}}{\partial u}, \frac{\partial \mathbf{F}}{\partial u} \right\rangle du^2 + 2 \left\langle \frac{\partial \mathbf{F}}{\partial u}, \frac{\partial \mathbf{F}}{\partial v} \right\rangle dudv + \left\langle \frac{\partial \mathbf{F}}{\partial v}, \frac{\partial \mathbf{F}}{\partial v} \right\rangle dv^2. \tag{2.5}$$

Let $N^{(l)} \in \mathcal{G}$, $l = 1, \ldots, n-2$ be the set of elements defined from

$$\langle N^{(l)}, N^{(l)} \rangle = 1, \quad \left\langle \frac{\partial \mathbf{F}}{\partial u}, N^{(l)} \right\rangle = \left\langle \frac{\partial \mathbf{F}}{\partial v}, N^{(l)} \right\rangle = 0, \tag{2.6}$$

$l = 1, \ldots, n-2$. The second fundamental form for $\mathbf{F}$ is defined to be

$$II = \left\langle \frac{\partial^2 \mathbf{F}}{\partial u^2}, N^{(l)} \right\rangle du^2 + 2 \left\langle \frac{\partial^2 \mathbf{F}}{\partial u \partial v}, N^{(l)} \right\rangle dudv + \left\langle \frac{\partial^2 \mathbf{F}}{\partial v^2}, N^{(l)} \right\rangle dv^2, \quad l = 1, \ldots, n-2. \tag{2.7}$$

The remarkable fact is that the surface in G can now be related to surfaces in its Lie algebra $\mathcal{G}$ by means of the adjoint representation. Using the adjoint representation, $\mathbf{F}$ and Φ are related by means of the system,

$$\frac{\partial \mathbf{F}}{\partial u} = \Phi^{-1} a_j e_j \Phi, \quad \frac{\partial \mathbf{F}}{\partial v} = \Phi^{-1} b_j e_j \Phi, \quad j = 1, \ldots, n, \tag{2.8}$$

and the a_j and b_j are functions of (u, v).