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Steffensen type inequalities for fuzzy integrals

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ABSTRACT

We provide several Steffensen type inequalities for the Sugeno integral. The inequalities are of the form

$$⨍_a^b fg\,\mathrm{d}\mu \leqslant A ⨍_a^{a+\lambda} f\,\mathrm{d}\mu + B \quad \text{or} \quad ⨍_a^b fg\,\mathrm{d}\mu \leqslant A ⨍_{b-\lambda}^{b} f\,\mathrm{d}\mu + B,$$

where A, B are constants, μ is a fuzzy measure on $\mathbb{R}$, g: $[a, b] \to [0, 1]$, $f : [a, b] \to \mathbb{R}_+$ is nonincreasing or nondecreasing and $\lambda = (b-a) \wedge ⨍_a^b g\,\mathrm{d}\mu$. We show that some of our sufficient conditions are also necessary.

1. Introduction

The theory of fuzzy measures and fuzzy integral was introduced by Sugeno [20] as a tool for modeling non-deterministic problems. The properties and applications of the Sugeno integral have been studied by many authors. In particular, Ralescu and Adams [12] investigated equivalent definitions of fuzzy integrals and Román-Flores et al. [13,15] examined level-continuity of fuzzy integrals and H-continuity of fuzzy measures. Wang and Klir [21] presented a general overview on fuzzy measurement and fuzzy integration theory.

Recently, the fuzzy integral counterparts of several classical inequalities, including Markov's, Chebyshev's, Jensen's, Minkowski's, Hölder's and Hardy's inequalities, are given by Flores-Franulič and Román-Flores [5,6], Mesiar and Ouyang [9], Román-Flores et al. [14,17], Agahi et al. [2], among others. In the literature, there is only one such extension of the Steffensen inequality (see [8]), invented by an actuary Steffensen for the study of the relationship between life annuities [19]. The inequality plays an important role in mathematical analysis (see [11,18,22]) as well as in other fields, e.g., it is used to estimate Chebyshev's functional, i.e., the difference between the integral of the product and the product of integrals (see [3]), and to evaluate bounds for expectations of order and record statistics (see [4,7]).

The purpose of this paper is to study Steffensen's inequality for the Sugeno integral. Some preliminaries are presented in Section 2. In Sections 3 and 4, we give several Steffensen type inequalities for nonincreasing and nondecreasing functions. Theorems 3.1 and 4.1 provide some sufficient and necessary conditions for our two inequalities, while Corollary 3.2 indicates a relationship between Steffensen's inequality and Chebyshev's inequality. Concluding remarks and open problems are formulated in Section 5.

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2. Preliminaries

In this section we recall some basic definitions and well-known results on Sugeno integral, which will be used in the next sections. We will denote by $\mathbb{R}_+$ the set $[0, \infty)$.

Definition 2.1. Let Σ be a σ-algebra of subsets of $\mathbb{R}$ and let $\mu\colon \Sigma \to [0, \infty]$ be a nonnegative, extended real-valued set function. We say that μ is a fuzzy measure if

(a) $\mu(\phi) = 0$,
(b) $E, F \in \Sigma$ and $E \subseteq F$ imply $\mu(E) \le \mu(F)$.

When μ is a fuzzy measure, the triple $(\mathbb{R}, \Sigma, \mu)$ is called a fuzzy measure space. In this paper, we assume that $\mu([a, b])$ is finite for all $a, b \in \mathbb{R}$ such that $a < b$. We do not assume continuity of fuzzy measures neither from below nor from above, in contrast to other works (cf. [1,5,16]).

For a fuzzy measure μ on $\mathbb{R}$, we define

$$\mathcal{F}^{\mu}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R}_+ : f \text{ is } \mu\text{-measurable}\}.$$

For $f \in \mathcal{F}^{\mu}(\mathbb{R})$, we will denote $\{x \in \mathbb{R} : f(x) \geqslant \alpha\}$ by $\{f \ge \alpha\}$. Clearly, $\{f \ge \beta\} \subseteq \{f \ge \alpha\}$ if $\alpha \le \beta$.

Definition 2.2. Let $(\mathbb{R}, \Sigma, \mu)$ be a fuzzy measure space and let $A \in \Sigma$. The Sugeno integral (or fuzzy integral) of f on A, with respect to the fuzzy measure μ, is defined (see [20]) as

$$⨍_A f \,\mathrm{d}\mu = \sup_{\alpha \geqslant 0}[\alpha \wedge \mu(A \cap \{f \geqslant \alpha\})],$$

where $x \wedge y = \min(x, y)$.

For simplicity of notation we write $\sup[\alpha \wedge f(\alpha)]$ instead of $\sup_{\alpha \geqslant 0}[\alpha \wedge f(\alpha)]$ and, on condition that f is defined on $[a, b]$, $\mu(\{f \ge \alpha\})$ instead of $\mu([a, b] \cap \{f \ge \alpha\})$.

Now we recall the basic properties of the Sugeno integral.

Theorem 2.1 ([21]). *Suppose μ is a fuzzy measure on $\mathbb{R}$, $A \subseteq \mathbb{R}$ and $f, g \in \mathcal{F}^{\mu}(\mathbb{R})$.*

(i) *If $f \le g$ on A, then $⨍_A f \,\mathrm{d}\mu \leqslant ⨍_A g \,\mathrm{d}\mu$.*
(ii) *For any $c \in [0, \infty)$, $⨍_A c \,\mathrm{d}\mu = c \wedge \mu(A)$.*
(iii) *If $f \le c$ on A, where $c \in [0, \infty)$, then $⨍_A f \,\mathrm{d}\mu \le c \wedge \mu(A)$.*

Proof. Conditions (i)–(ii) follow directly from the definition of the Sugeno integral. Condition (iii) is a straightforward consequence of conditions (i)–(ii). □

3. The Steffensen inequality for nonincreasing functions

The classical Steffensen integral inequalities are of the form

$$\int_{b-\lambda}^{b} f(t)\,\mathrm{d}t \leqslant \int_a^b f(t)g(t)\,\mathrm{d}t \leqslant \int_a^{a+\lambda} f(t)\,\mathrm{d}t, \tag{1}$$

where f and g defined on $[a, b]$ are Riemann integrable functions, f is nonincreasing, $0 \le g(t) \le 1$ for $t \in (a, b)$ and $\lambda = \int_a^b g(t)\,\mathrm{d}t$ (see [10,19]). In this paper we deal with the study of the upper Steffensen inequality.

The following example shows that the Steffensen inequality for the Sugeno integral is not valid for nonincreasing $f \in \mathcal{F}^{\mu}([a, b])$ and for $g\colon [a, b] \to [0, 1]$.

Example 3.1. Let $f(x) = 2 - x$, $g(x) = 0.5$, $x \in [0, 1]$ and let μ be the Lebesgue measure on $\mathbb{R}$. It is easily seen that f and g satisfy the assumptions of the classical Steffensen inequality. A straightforward calculus shows that:

$$⨍_0^1 fg \,\mathrm{d}\mu = 2/3, \quad \lambda = ⨍_0^1 g \,\mathrm{d}\mu = 0.5, \quad ⨍_0^{\lambda} f \,\mathrm{d}\mu = 0.5.$$

In consequence,

$$2/3 = ⨍_0^1 fg \,\mathrm{d}\mu > ⨍_0^{\lambda} f \,\mathrm{d}\mu = 0.5,$$

which implies that inequality (1) is not satisfied for the Sugeno integral.