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# Connection formulas for general discrete Sobolev polynomials: Mehler–Heine asymptotics ☆

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ABSTRACT

In this paper the discrete Sobolev inner product

$$\langle p, q\rangle = \int p(x)q(x)\,d\mu + \sum_{i=0}^{r} M_i\, p^{(i)}(c)\, q^{(i)}(c)$$

is considered, where $\mu$ is a finite positive Borel measure supported on an infinite subset of the real line, $c \in \mathbb{R}$ and $M_i \geq 0, i = 0, 1, \ldots, r$.

Connection formulas for the orthonormal polynomials associated with $\langle ., .\rangle$ are obtained. As a consequence, for a wide class of measures $\mu$, we give the Mehler–Heine asymptotics in the case of the point $c$ is a hard edge of the support of $\mu$. In particular, the case of a symmetric measure $\mu$ is analyzed. Finally, some examples are presented.



## 1. Introduction

Let $\{p_n(x)\}_{n\geq 0}$ be the sequence of orthonormal polynomials with respect to a finite positive Borel measure $\mu$, supported on an infinite subset of $\mathbb{R}$. We denote by $\{q_n(x)\}_{n\geq 0}$ the sequence of orthonormal polynomials with respect to an inner product of the form

$$\langle p, q\rangle = \int p(x)q(x)\,d\mu + \sum_{i=0}^{r} M_i\, p^{(i)}(c)\, q^{(i)}(c), \qquad c \in \mathbb{R} \tag{1}$$

where $M_i \geq 0, i = 0, \ldots, r-1$ and $M_r > 0$. Such inner products are called discrete Sobolev or Sobolev type and they have been considered in different contexts.

In this paper we focus our attention on Mehler–Heine asymptotics for discrete Sobolev orthogonal polynomials. This asymptotic gives us one of the main differences that can be established in order to show how the addition of the derivatives in the inner product influences the orthogonal system. The Mehler–Heine formulas describe the asymptotic behavior of orthogonal polynomials near the hard edge, i.e. those endpoints of the support of the zero distribution which are also endpoints of the support of the measure. Thus, our interest is to show how the presence of the masses in the inner product changes the asymptotic behavior around this point.

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To prove the Mehler–Heine formula for Jacobi and Laguerre polynomials, usually the explicit representation of these polynomials is used (see [18]). Although the situation is a bit different, we would like to mention that recently, in [19], for some classical multiple orthogonal polynomials, asymptotic formulas of Mehler–Heine type are obtained using the explicit expression of the polynomials and the Lebesgue's dominated convergence theorem. But there are many other polynomials for which we have not an explicit representation. For example, Aptekarev in [5] realizes that for certain classes of weight functions supported on $[-1, 1]$, the Mehler–Heine asymptotic formula depends on the local behavior at the endpoint of the interval of orthogonality. So, this formula has been extended to a broader class of measures belonging to the Nevai's class. This result has been applied to deduce the Mehler–Heine formula for the generalized Jacobi polynomials (see [7]). On the other hand, for exponential weights (see [3]), it has been proved the Mehler–Heine formula for the so-called Freud polynomials using the asymptotic formula given by Kriecherbauer and McLaughlin in [12] obtained by the Riemann–Hilbert method.

For discrete Sobolev orthogonal polynomials, it is difficult to apply the same method as Jacobi and Laguerre. This analytic idea was developed for the discrete Laguerre Sobolev orthogonal polynomials in [2]. There, the authors obtained a new and specific formula for the derivatives of $q_n$ which leads to achieve a uniform bound in order to use the Lebesgue's dominated convergence theorem. But in a general case, this is quite complicated.

On the other hand, an important tool to get asymptotics is the knowledge of certain connection formulas for $q_n$ in terms of standard polynomials related with $p_n$.

One of them can be deduced from the well known fact that $\{q_n(x)\}_{n\geq 0}$ is quasi-orthogonal of order $r+1$ and consequently we can express $q_n$ as a linear combination of the standard orthogonal polynomials corresponding to the modified measure $(x-c)^{r+1}\,d\mu(x)$. This connection formula has proved to be fruitful, for example, in the study of relative asymptotics when $\mu$ has compact support, see [13] and [17] in a more general setting. However, the situation is quite different in the case of measures with unbounded support. So, in [2], it was shown that this connection formula is not the adequate to study neither relative asymptotics nor Mehler–Heine formula when $\mu$ is the Laguerre weight.

For discrete Laguerre Sobolev orthogonal polynomials, another connection formula was given by Koekoek in [10] with an arbitrary $r$ and $c = 0$. This formula has turned out to be of great importance to generating Mehler–Heine asymptotics. So, in [4], using the explicit expression of the connection coefficients given in [11] for $r = 1$, the authors prove the Mehler–Heine asymptotic for the corresponding Laguerre Sobolev polynomials. This idea has also been used in [14] for $r = 1$ and $c < 0$. However, in an inner product with an arbitrary (finite) number of terms in the discrete part, the problem is that we have not the explicit expression of the coefficients. In spite of this, in [16] the authors achieve the Mehler–Heine formula for an arbitrary number of masses and $c = 0$.

In this paper we prove that for a wide class of measures with support bounded or not, an arbitrary number of masses in the inner product (1) and without taking into account the location of the point $c$ with respect to the support of $\mu$, there exists a connection formula for $q_n(x)$ in terms of some canonical transformation of the polynomials $p_n$, called Christoffel perturbations. More precisely, $q_n(x) = \sum_{j=0}^{r+1} \lambda_{j,n}(x-c)^j\, p_{n-j}^{[2j]}(x)$ where $\{p_n^{[j]}(x)\}_{n\geq 0}$ denotes the sequence of orthonormal polynomials with respect to the measure $\mu_j$ with $d\mu_j(x) = (x-c)^j\,d\mu(x)$. The main contribution is that we are able to give information of asymptotic behavior of the connection coefficients, without the explicit expression of them. This is a significant improvement compared with the previous works. Our interest is focused on the application of this connection formula for obtaining the Mehler–Heine asymptotics and so to prove that whenever the asymptotic behavior near the hard edge involves Bessel functions, the presence of positive masses in the inner product produces a convergence acceleration to this point of $r+1$ zeros of the Sobolev polynomials.

Finally, we would like to notice that the connection formula obtained has interest by itself because it may be used to get other results as Cohen type inequality and other asymptotic properties.

The structure of the paper is as follows. In Section 2, we establish a connection formula for orthonormal polynomials with respect to the inner product (1), where $c$ is an arbitrary real number. Moreover, we show a technical lemma that besides giving us asymptotic behavior at the point $c$ of the successive derivatives of the polynomials $q_n$ and so of their kernels, also provides information about the asymptotic behavior of the coefficients in the connection formula. In Section 3, for a wide class of measures $\mu$, we obtain Mehler–Heine asymptotics for the sequence $\{q_n(x)\}_{n\geq 0}$ where in (1) all the masses are positive and the point $c$ is either an endpoint of the interval where the measure $\mu$ is supported or the origin if the measure is symmetric. As an application, we obtain an important information about the distribution of the zeros of the polynomials $q_n$. In the last section we present some examples to illustrate the theory given.

Throughout this paper we use the notation $x_n \cong y_n$ when the sequence $x_n/y_n$ converges to 1.

## 2. Connection formulas

Let $\{p_n(x)\}_{n\geq 0}$ be the sequence of orthonormal polynomials with respect to the measure $\mu$ and $\{q_n(x)\}_{n\geq 0}$ the sequence of orthonormal polynomials with respect to the inner product (1).

In this section we will establish a connection formula for the discrete Sobolev polynomials $q_n$ in terms of the polynomials $p_n^{[2j]}(x)$ orthogonal with respect to the measure $d\mu_{2j}(x) = (x-c)^{2j}\,d\mu(x)$.

**Theorem 1.** *Assume that the polynomials* $\{p_n(x)\}_{n\geq 0}$ *satisfy*

$$p_n(c)\,p_{n-1}^{[2]}(c)\,\ldots\,p_{n-(r+1)}^{[2(r+1)]}(c) \neq 0 \tag{2}$$