Applied Mathematics and Computation 261 (2015) 242–251

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Approximations of solutions for a nonlinear differential equation with a deviating argument

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ARTICLE INFO

MSC:
34K30
34G20
47H06

Keywords:
Differential equation with deviated argument
Banach fixed point theorem
Analytic semigroup

ABSTRACT

In this paper, we prove the existence and convergence of approximate solution for a class of nonlinear differential equations with a deviated argument in a Hilbert space. We establish the existence and uniqueness of a solution to every approximate integral equation using the fixed point argument. Then, we prove the convergence of a solution of the approximate integral equation to the solution of the associated integral equation. We also consider the Faedo–Galerkin approximation of a solution and prove some convergence results.

1. Introduction

Differential equations with a deviating argument have many applications such as in theory of automatic control, theory of self-oscillating systems, problems of long-term planning in economics, study of problems related with combustion in rocket motion, a series of biological problems, and many other areas of science and technology, the number is steadily expanding; for more details and for a good introduction, we refer to [6]. The question of existence and uniqueness of a solution for differential equations with a deviating argument had studied by many authors in recent years, see e.g. [7,8,9] and references cited therein.

Our objective in this paper is to study the following nonlinear differential equation with a deviating argument in a separable Hilbert space H:

$$\left.\begin{aligned}\frac{d}{dt}u(t)+Au(t)&=f(t,u(t),u(h(u(t),t))),\ 0<t\le T<\infty;\\ u(0)&=u_0,\end{aligned}\right\}\tag{1.1}$$

where $A: D(A)\subset H\to H$ is a closed, densely defined, positive definite and self adjoint linear operator. The functions $f:[0,T]\times H\times H\to H$ and $h: H\times[0,T]\to\mathbb{R}$ satisfy some conditions mentioned in Section 2.

Gal [7] has studied the existence and uniqueness of a solution to problem (1.1) where $-A$ is the infinitesimal generator of an analytic semigroup of bounded linear operators on a Banach space. The nonlinear H-valued function $f(t,x,y)$ is uniformly locally Hölder continuous in t, uniformly locally Lipschitz continuous in x with respect to the X_α-norm, the domain of the α-power of A; uniformly locally Lipschitz continuous in y with respect to the $X_{\alpha-1}$-norm; and the function $h(x,t)$ is uniformly locally Hölder continuous in t, uniformly locally Lipschitz continuous in x with respect to the X_α-norm (see Section 2).

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http://dx.doi.org/10.1016/j.amc.2015.03.071
0096-3003/

Milleta [10] have discusssed the Faedo–Galerkin approximation of a solution to the particular case of (1.1) where $h \equiv 0$ and $f(t, u) = M(u)$. The more general case has discussed in [2]. For more details on existence of approximate solutions and related study of various problems, we refer to [1–5,10,11].

In the present work, we are interested in the Faedo–Galerkin approximation of a solution to problem (1.1). In Section 2, we provide some of the notations, notions and results that will be required in later sections. In Section 3, we consider an integral equation associated with (1.1) and then consider a sequence of approximate integral equations and establish the existence and uniqueness of a solution for each of the approximate integral equation. Also we prove the convergence of the solution of the approximate integral equation in Section 4. In Section 5, we consider the Faedo–Galerkin approximation of a solution and prove some convergence results for such an approximation. Finally, we have given an example to show applications of abstract results obtained in the earlier sections.

The results presented in this paper can be applied easily to a problem of the type (1.1) with a nonlocal condition under some modified assumptions on the function f and the operator A.

2. Preliminaries and assumptions

In this section, we present assumptions, preliminaries and lemmas that are required for proving our main results. The material presented here can be found in more details in [12]. We shall use the following assumption on the operator A:

(H1) Let A be a closed, positive definite, self-adjoint linear operator from the domain $D(A) \subset H$ into H with $D(A)$ is dense in H. We also assume that A has the pure point spectrum

$$0 < \lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_m \le \cdots,$$

where $\lambda_m \to \infty$ as $m \to \infty$ and a corresponding complete orthonormal system of eigenfunctions $\{\phi_i\}$, i.e.,

$$A\phi_i = \lambda_i \phi_i \quad \text{and} \quad \langle \phi_i, \phi_j \rangle = \delta_{ij},$$

where $\delta_{ij} = 1$ if $i = j$ and zero otherwise.

These assumptions on A guarantee that $-A$ generates an analytic semigroup of bounded operators, denoted by $S(t)$, $t \ge 0$. It is known that there exist constants $\tilde{M} \ge 1$ and $\omega \ge 0$ such that

$$\|S(t)\| \le \tilde{M} e^{\omega t}, \quad t \ge 0.$$

Since $-A$ generates the analytic semigroup $S(t)$, $t \ge 0$, we may add cI to $-A$ for some constant c, if necessary, we may assume without loss of generality that $\|S(t)\|$ is uniformly bounded by M, i.e., $\|S(t)\| \le M$ and $0 \in \rho(A)$. In this case, it is possible to define the fractional power A^α for $0 \le \alpha \le 1$ as closed linear operator with domain $D(A^\alpha) \subseteq H$. Furthermore, $D(A^\alpha)$ is dense in H and the expression

$$\|x\|_\alpha = \|A^\alpha x\|,$$

defines a norm on $D(A^\alpha)$. Henceforth, we denote the space $D(A^\alpha)$ by H_α endowed with the norm $\|\cdot\|_\alpha$. Also, for each $\alpha > 0$, we define $H_{-\alpha} = (H_\alpha)^*$, the dual space of H_α, is a Hilbert space endowed with the norm $\|x\|_{-\alpha} = \|A^{-\alpha} x\|$.

The following results can be obtained, as a consequence of assumptions on A, for an analytic semigroup $S(t)$, $t \ge 0$ (see [12, pp. 195–196]).

Lemma 2.1. *Suppose that $-A$ is the infinitesimal generator of an analytic semigroup $S(t)$, $t \ge 0$ with $\|S(t)\| \le M$ for $t \ge 0$ and $0 \in \rho(-A)$. Then we have the following properties:*

(i) *H_α is a Hilbert space for $0 \le \alpha \le 1$;*
(ii) *For $0 < \delta \le \alpha < 1$, the embedding $H_\alpha \hookrightarrow H_\delta$ is continuous;*
(iii) *A^α commutes with $S(t)$ and there exists a constant $C_\alpha > 0$ depending on $0 \le \alpha \le 1$ such that*

$$\|A^\alpha S(t)\| \le C_\alpha t^{-\alpha}, \quad t > 0.$$

For more details on the fractional powers of closed linear operators we refer to Pazy [12].

We denote the space of all H_α-valued continuous functions in [0,t], for all $t \in [0, T]$, by $\mathcal{C}_t^\alpha$. Thus $\mathcal{C}_t^\alpha = C([0, t]; H_\alpha)$, for all $t \in [0, T]$, is a Banach space endowed with the supremum norm,

$$\|\psi\|_{t,\alpha} := \sup_{0 \le \eta \le t} \|\psi(\eta)\|_\alpha, \quad \psi \in \mathcal{C}_t^\alpha.$$

For $0 \le \alpha < 1$, define

$$\mathcal{C}_T^{\alpha-1} = \{y \in \mathcal{C}_T^\alpha : \|y(t) - y(s)\|_{\alpha-1} \le L|t-s|, \forall\ t, s \in [0, T]\},$$

where L is a suitable positive constant to be specified later.

We shall use the following condition on f and h in its arguments: