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Efficient computation of highly oscillatory integrals with Hankel kernel ☆

Zhenhua Xu [a], Gradimir V. Milovanović [b], Shuhuang Xiang [a,*]

[a] School of Mathematics and Statistics, Central South University, Changsha, Hunan 410083, China
[b] Serbian Academy of Sciences and Arts, Beograd, Serbia & State University of Novi Pazar, Serbia

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ABSTRACT

In this paper, we consider the evaluation of two kinds of oscillatory integrals with a Hankel function as kernel. We first rewrite these integrals as the integrals of Fourier-type. By analytic continuation, these Fourier-type integrals can be transformed into the integrals on $[0, +\infty)$, the integrands of which are not oscillatory, and decay exponentially fast. Consequently, the transformed integrals can be efficiently computed by using the generalized Gauss–Laguerre quadrature rule. Moreover, the error analysis for the presented methods is given. The efficiency and accuracy of the methods have been demonstrated by both numerical experiments and theoretical results.

1. Introduction

In this paper, we are concerned with the numerical approximation of the integrals with a highly oscillatory Hankel kernel of the form

$$I_1[f] = \int_a^b f(x)H_\nu^{(1)}(\omega x)\mathrm{d}x \quad \text{and} \quad I_2[f] = \int_a^{+\infty} f(x)H_\nu^{(1)}(\omega x)\mathrm{d}x, \tag{1.1}$$

where $H_\nu^{(1)}(x) = J_\nu(x) + \mathrm{i}Y_\nu(x)$ is Hankel function of the first kind of order ν, $\omega \gg 1$ and $b > a > 0$. The integrals (1.1) play an important role in many areas of science and engineering, such as astronomy, optics, quantum mechanics, seismology image processing, electromagnetic scattering, such as [3,4,11,17].

For large values of ω, the integrands become highly oscillatory, and the efficient and reliable numerical evaluation of the integrals is problematic. Moreover, a prohibitively number of quadrature nodes are needed to obtain satisfied accuracy if one uses classical numerical methods like Simpson rule, Gaussian quadrature, etc. In the last few years, many efficient methods have been devised for the integral $\int_a^b f(x)H_\nu^{(1)}(\omega x)\mathrm{d}x$, such as Levin method [20,21], Levin-type method [25], modified Clenshaw–Curtis method [26], generalized quadrature rule [12,13], Filon-type method [30], Clenshaw–Curtis–Filon-type method [31], Gauss–Laguerre quadrature [5,6]. However, only a few methods to evaluate the integral $\int_a^\infty f(x)H_\nu^{(1)}(\omega x)\mathrm{d}x$ have been proposed. For the latest references, we refer the readers to [7,8] for a more general review.

All above-mentioned methods share the property that the larger the ω, the higher the accuracy. The goal of this paper is to explore efficient and high order methods for the integrals (1.1) based on the idea of complex integration method (see

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* Corresponding author. Tel.: +86 15116342394.
E-mail addresses: xuzhenhua19860536@163.com (Z. Xu), gvm@mi.sanu.ac.rs (G.V. Milovanović), xiangsh@mail.csu.edu.cn (S. Xiang).

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Milovanović [23], Huybrechs and Vandewalle [16] and Chen [5,6]), which can also be applied to the computation of highly oscillatory Cauchy principal value integrals and highly oscillatory integrals with algebraic singularities [18,19,27,28,32].

An outline of this paper is as follows. In Section 2, we present the details of the proposed methods for computing the integrals (1.1). Meanwhile, error analysis for the presented methods is discussed. In Section 3, several numerical examples are given to show the efficiency and accuracy of presented methods.

2. Numerical schemes for the integrals

Thanks to the identity [15, p. 915]

$$H_\nu^{(1)}(\omega x) = \sqrt{\frac{2}{\pi\omega x}}\frac{e^{i(\omega x - \frac{\pi}{2}\nu - \frac{\pi}{4})}}{\Gamma(\nu+\frac{1}{2})}\int_0^{+\infty}\left(1+\frac{it}{2\omega x}\right)^{\nu-\frac{1}{2}} t^{\nu-\frac{1}{2}}e^{-t}dt, \quad x>0, \tag{2.1}$$

we can rewrite the integrals (1.1) in the following form

$$\begin{aligned} I_1[f] &= \sqrt{\frac{2}{\pi\omega}}\frac{e^{-i\pi(2\nu+1)/4}}{\Gamma(\nu+\frac{1}{2})}\int_a^b f(x)x^{-\frac{1}{2}}e^{i\omega x}\left[\int_0^{+\infty}\left(1+\frac{it}{2\omega x}\right)^{\nu-\frac{1}{2}} t^{\nu-\frac{1}{2}}e^{-t}dt\right]dx \\ &= \sqrt{\frac{2}{\pi\omega}}\frac{e^{-i\pi(2\nu+1)/4}}{\Gamma(\nu+\frac{1}{2})}\int_a^b f(x)x^{-\frac{1}{2}}g(x)e^{i\omega x}dx, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} I_2[f] &= \sqrt{\frac{2}{\pi\omega}}\frac{e^{-i\pi(2\nu+1)/4}}{\Gamma(\nu+\frac{1}{2})}\int_a^{+\infty} f(x)x^{-\frac{1}{2}}e^{i\omega x}\left[\int_0^{+\infty}\left(1+\frac{it}{2\omega x}\right)^{\nu-\frac{1}{2}} t^{\nu-\frac{1}{2}}e^{-t}dt\right]dx \\ &= \sqrt{\frac{2}{\pi\omega}}\frac{e^{-i\pi(2\nu+1)/4}}{\Gamma(\nu+\frac{1}{2})}\int_a^{+\infty} f(x)x^{-\frac{1}{2}}g(x)e^{i\omega x}dx, \end{aligned} \tag{2.3}$$

where

$$g(x) = \int_0^{+\infty}\left(1+\frac{it}{2\omega x}\right)^{\nu-\frac{1}{2}} t^{\nu-\frac{1}{2}}e^{-t}dt. \tag{2.4}$$

From Eqs. (2.2) and (2.3), we can see that the integrals (1.1) are transformed into the integrals of Fourier form, which can be evaluated by complex integration method and quadrature rules of Gaussian type. In what follows, we will focus on the fast computation of the integrals (1.1) and error analysis for the presented methods.

2.1. The evaluation of the integral $I_1[f]$

For the calculation of the integral $I_1[f]$, we first assume that f is an analytic function in the half-strip of the complex plane, $a \le \mathrm{Re}(z) \le b$, $\mathrm{Im}(z) \ge 0$, and there exists two constants C and ω_0, such that $|f(x+iR)| \le Ce^{\omega_0 R}$, $a \le x \le b$, with $0 < \omega_0 < \omega$.

Following the ideas of [5,16,23], consider the contour as shown in Fig. 1 for the integral (2.2), and let D denote the region

$$D = \{z \in \mathbb{C} \mid a \le \mathrm{Re}(z) \le b,\ 0 \le \mathrm{Im}(z) \le R\}.$$

Since the integrand

$$F(z) = f(z)z^{-\frac{1}{2}}g(z) \tag{2.5}$$

is analytic in the region D, by the Cauchy Residue Theorem [1], we have

$$\int_{\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4} F(z)e^{i\omega z}dz = 0, \tag{2.6}$$

with all the contours taken in the counterclockwise direction.

For the integral over the contour Γ_2, we have

$$\begin{aligned} \int_{\Gamma_2} F(z)e^{i\omega z}dz &= i\int_0^R F(b+ip)e^{i\omega(b+ip)}dp \\ &= ie^{i\omega b}\int_0^R F(b+ip)e^{-\omega p}dp \\ &= \frac{ie^{i\omega b}}{\omega}\int_0^{\omega R} F\left(b+\frac{ip}{\omega}\right)e^{-p}dp. \end{aligned} \tag{2.7}$$