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Inequalities and asymptotic expansions associated with the Ramanujan and Nemes formulas for the gamma function

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A B S T R A C T

We present some inequalities and asymptotic expansions associated with the Ramanujan and Nemes formulas for the gamma function.

1. Introduction

The Indian mathematician Ramanujan (see [25, p. 339] and [4, pp. 117–118]) claimed that

$$\Gamma(x+1) = \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3+4x^2+x+\frac{\theta_x}{30}\right)^{1/6},$$

where $\theta_x \to 1$ as $x \to \infty$ and $\frac{3}{10} < \theta_x < 1$. That is,

$$\Gamma(x+1) \approx \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3+4x^2+x+\frac{1}{30}\right)^{1/6}, \quad x \to \infty \tag{1.1}$$

and

$$\sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3+4x^2+x+\frac{1}{100}\right)^{1/6} < \Gamma(x+1) < \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3+4x^2+x+\frac{1}{30}\right)^{1/6}, \quad x \geq 0.$$

Ramanujan's claim has been the subject of intense investigations and is reviewed in [5, p. 48, Question 754], and has motivated a large number of research papers (see, for example, [3,6,7,10,12–14,17–19,21–23,26]).

Karatsuba [14] proved that the function

$$h(x) := \left[\left(\frac{e}{x}\right)^x \frac{\Gamma(x+1)}{\sqrt{\pi}}\right]^6 - (8x^3+4x^2+x) = \frac{\theta_x}{30}$$

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is monotonically increasing from $[1, \infty)$ onto $[h(1), h(\infty))$ with $h(1) = e^6/\pi^3 - 13 = 0.0111976\ldots$ and $h(\infty) = 1/30 = 0.0333\ldots$. Also, Karatsuba [14, Eq. (5.5)] established the asymptotic representation of the gamma function

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{1}{30} - \frac{11}{240x} + \frac{79}{3360x^2} + \frac{3539}{201600x^3} - \frac{9511}{403200x^4} - \frac{10051}{716800x^5} + \frac{233934691}{6386688000x^6} + \cdots\right)^{1/6}, \quad x \to \infty \tag{1.2}$$

(x^{-6} term corrected). Moreover, the author gave a formula for successively determining the coefficients. Alzer [3] proved that in $(0, 1]$ the constant term $\frac{1}{100}$ can be replaced by the best possible $\min_{0.6 \le x \le 0.7} \theta_x = 0.0100450\ldots$ and that the improved double inequality for θ_x holds for $0 \le x < \infty$.

Hirschhorn [12] established that θ_n (for $n \in \mathbb{N} := \{1, 2, \ldots\}$) satisfies the inequalities

$$1 - \frac{11}{8n} + \frac{5}{8n^2} < \theta_n < 1 - \frac{11}{8n} + \frac{11}{8n^2}. \tag{1.3}$$

In [26], a proof of the monotonicity of θ_n was given and the concavity of θ_n was also noted without proof. Hirschhorn and Villarino [13] improved (1.3) and obtained the following inequality

$$1 - \frac{11}{8n} + \frac{79}{112n^2} < \theta_n < 1 - \frac{11}{8n} + \frac{79}{112n^2} + \frac{20}{33n^3}, \quad n \in \mathbb{N}, \tag{1.4}$$

and then used it to prove that θ_n is increasing and concave.

Recently, Mortici [17] presented the following analogous result to (1.1):

$$\Gamma(x+1) \approx \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(16x^4 + \frac{32}{3}x^3 + \frac{32}{9}x^2 + \frac{176}{405}x - \frac{128}{1215}\right)^{1/8}, \quad x \to \infty. \tag{1.5}$$

The first aim of present paper is to unify the formulas (1.2) and (1.5), and develop (1.5) to produce a complete asymptotic expansion. More precisely, we give a recursive relation for determining the coefficients $p_j \equiv p_j(r)$ such that

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(2^r x^r + \sum_{j=1}^{\infty} p_j x^{r-j}\right)^{1/(2r)}, \quad x \to \infty,$$

where $r \neq 0$ is a given real number.

Very recently, Chen [7] presented a sharp version of Ramanujan's inequality for the factorial function and obtained the following result. For all $n \in \mathbb{N}$,

$$\sqrt{\pi}\left(\frac{n}{e}\right)^n \left(8n^3 + 4n^2 + n + \frac{1}{30}\right)^{1/6} \left(1 - \frac{11}{11520(n+a)^4}\right) < \Gamma(n+1) \le \sqrt{\pi}\left(\frac{n}{e}\right)^n \left(8n^3 + 4n^2 + n + \frac{1}{30}\right)^{1/6} \left(1 - \frac{11}{11520(n+b)^4}\right) \tag{1.6}$$

with the best possible constants

$$a = \frac{39}{154} = 0.2532467532\ldots$$

and

$$b = \left(\frac{11}{11520\left(1 - \left(\frac{30e^6}{391\pi^3}\right)^{\frac{1}{6}}\right)}\right)^{\frac{1}{4}} - 1 = 0.3549912666\ldots.$$

Using the Maple software, we find, as $x \to \infty$,

$$\frac{\Gamma(x+1)}{\sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{1}{30}\right)^{1/6}} = 1 - \frac{\frac{11}{11520}}{(x + \frac{39}{154})^4} + O\left(\frac{1}{x^6}\right) \tag{1.7}$$

and

$$\frac{\Gamma(x+1)}{\sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{1}{30}\right)^{1/6}} = 1 - \frac{\frac{11}{11520}}{x^4 + \frac{78}{77}x^3 + \frac{365579}{355740}x^2 + \frac{11084441}{27391980}x - \frac{308425057271}{1349876774400}} + O\left(\frac{1}{x^9}\right). \tag{1.8}$$

Obviously, the formula (1.8) is better than the formulas (1.7).