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Positive solutions of sub-superlinear Sturm–Liouville problems

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ABSTRACT

In this paper, we introduce the notion of a strict lower/upper solution to nonlinear Sturm–Liouville boundary value problems. Based on the maximum principles, we establish a result of Leray–Schauder degree on the ordered intervals induced by the pairs of strict lower and upper solutions. Applying the result and the fixed point index theory in cones, we obtain the global existence results of positive solutions for sub-superlinear Sturm–Liouville problems.

1. Introduction

In this paper, we are concerned with the global existence of positive solutions to the following Sturm–Liouville problem

$$\begin{cases} -(p(t)u')' + q(t)u = \lambda f(t,u), \ t \in (0,1), \\ au(0) - bp(0)u'(0) = 0, \\ cu(1) + dp(1)u'(1) = 0 \end{cases} \tag{1.1}$$

with respect to positive parameter λ, where functions p, q, f and constants a, b, c, d satisfy the following conditions:

(S_1) $p \in C^1[0,1]$, $q \in C[0,1]$, and $p(t) > 0$, $q(t) \geq 0$, $\forall t \in [0,1]$;
(S_2) $a, b, c, d \geq 0$ and $(a+b)(c+d) > 0$, moreover $q(t) \not\equiv 0$ if $a = c = 0$;
(S_3) $f \in C([0,1] \times \mathbb{R}^+, \mathbb{R}_0^+)$, here $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{R}_0^+ = (0, +\infty)$.

Problem (1.1) arises in many different areas of applied mathematics and physics, and only its positive solution is significant in some practice (see [1,4,7,15,16,21,28]).

For problem (1.1) with $\lambda = 1$, $p(t) \equiv 1$ and $q(t) \equiv 0$, that is, the following problem

$$\begin{cases} -u'' = f(t,u), \ t \in (0,1), \\ au(0) - bu'(0) = 0, \\ cu(1) + du'(1) = 0, \end{cases} \tag{1.2}$$

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the existence of positive solutions has been studied by many authors. It is clear that a necessary condition for the existence of positive solutions to problem (1.2) is

$$\inf_{s\in\mathbb{R}^+}\frac{f(t,s)}{s}\leqslant\delta_1\leqslant\sup_{s\in\mathbb{R}^+}\frac{f(t,s)}{s},\tag{1.3}$$

here δ_1 is the first eigenvalue of the corresponding linear eigenvalue problem. Hence, nonlinearity f is usually imposed on some conditions involving the limit behavior of $f(t,s)/s$ at zero and at infinity. The sublinear case, i.e., the case where $\lim_{s\to 0}f(t,s)/s=\infty$ and $\lim_{s\to\infty}f(t,s)/s=0$, was considered by the upper and lower solutions method, such as in [13,16,17,28], but subjected to Dirichlet boundary conditions. Subsequently, in [19,20,22,36] they dealt with the general separated boundary value problem by use of the fixed point index theory in cones in the case that f is nonnegative and continuous. Based on some conditions involving the first eigenvalue of the corresponding linear eigenvalue problem, other related results were given by the authors in [19,20,25,30,32,36]. For the superlinear case, i.e., the case where $\lim_{s\to 0}f(t,s)/s=0$ and $\lim_{s\to\infty}f(t,s)/s=\infty$, it is usual to apply the topological degree theory, especially, the fixed point index theory in cones, see [9,16,17,19,25,30,32,36] and references therein. In [18], the authors considered the sub-superlinear case, i.e., the case $\lim_{s\to 0}f(t,s)/s=\infty$ and $\lim_{s\to\infty}f(t,s)/s=\infty$, for an elliptic Dirichlet problem. As seen from (1.3) these conditions are not enough for the existence of positive solutions, therefore they made the additional assumption that there is a strict upper solution and proved the existence of two positive solutions. An alternative assumption was introduced in [12], which implies the existence of a strict upper solution in the particular case of an ordinary differential equation. The super-sublinear case, i.e., the case $\lim_{s\to 0}f(t,s)/s=\lim_{s\to\infty}f(t,s)/s=0$, was considered in [33], see also [19] for problem (1.2). For Sturm–Liouville boundary value problem (1.1) with $\lambda=1$, the authors in [26,27,34,35] considered the general sublinear case and superlinear case involving the principal eigenvalue of Sturm–Liouville operator, in some sense their conditions are optimal. Recently, in [5,8,24,31] Sturm–Liouville boundary value problems have been studied by use of global bifurcation theory, variational methods and topological methods.

To the best of our knowledge, Sturm–Liouville nonlinear eigenvalue problems are less discussed. For problem (1.1) with the special case $p(t)\equiv 1$ and $q(t)\equiv 0$, several authors dealt with the global existence of positive solutions, see [11,23,29] and the references therein. In particular, in view of [[27], Theorem 1] it is easy to see that for any positive parameter λ problem (1.1) has positive solutions when f is superlinear (i.e., $\lim_{s\to 0}f(t,s)/s=0$ and $\lim_{s\to\infty}f(t,s)/s=\infty$) or sublinear (i.e., $\lim_{s\to 0}f(t,s)/s=\infty$ and $\lim_{s\to\infty}f(t,s)/s=0$). Naturally, an interesting problem is whether problem (1.1) has positive solutions as f is sub-superlinear.

In this paper, our aim is to deal with the global existence of positive solutions for problem (1.1) with sub-superlinear terms. Motivated by some ideas in [2–4,6,10,14], in Section 2 we introduce the notion of a strict lower/upper solution and establish a result of Leray–Schauder degree on the ordered intervals induced by the pairs of strict lower and upper solutions. Applying the result and the fixed point index theory in cones, in Section 3 we obtain the global existence results of positive solutions for problem (1.1) with the sub-superlinear terms.

2. Sub-supersolution and Leray–Schauder degree

Let us consider

$$\begin{cases}-(p(t)u')'+q(t)u=F(t,u), & t\in(0,1),\\ au(0)-bp(0)u'(0)=0,\\ cu(1)+dp(1)u'(1)=0,\end{cases}\tag{2.1}$$

where $F:\mathscr{D}\to\mathbb{R}$ is a continuous function and $\mathscr{D}\subset[0,1]\times\mathbb{R}$.

Definition 2.1. For $\alpha\in C^2(0,1)\cap C^1[0,1]$, α is said to be a strict lower solution of (2.1) if $(t,\alpha(t))\in\mathscr{D}$ for all $t\in(0,1)$ and

$$\begin{cases}-(p(t)\alpha')'+q(t)\alpha-F(t,\alpha)<0, & t\in(0,1),\\ a\alpha(0)-bp(0)\alpha'(0)<0,\\ c\alpha(1)+dp(1)\alpha'(1)<0.\end{cases}\tag{2.2}$$

A strict upper solution $\beta\in C^2(0,1)\cap C^1[0,1]$ can also be defined if it satisfies the reverse of the above inequalities.

Lemma 2.1. [7,15] *Let X be a real Banach space and $A: X\to X$ be compactly continuous. For $r>0$, denote $B_r=\{x\in X: \|x\|<r\}$ and $\overline{B}_r=\{x\in X:\|x\|\leqslant r\}$. Assume that $A:\overline{B}_r\to B_r$, then*

$$\deg(id-A,B_r,\theta)=1.$$

Next, we consider the fixed point operator associated with (2.1), i.e., the compact operator $T: C[0,1]\to C[0,1]$ defined by $T\varphi\overset{\text{def}}{=}u$ if

$$\begin{cases}-(p(t)u')'+q(t)u=F(t,\varphi), & t\in(0,1),\\ au(0)-bp(0)u'(0)=0,\\ cu(1)+dp(1)u'(1)=0.\end{cases}\tag{2.3}$$