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Barnes-type degenerate Euler polynomials

Dmitry V. Dolgy[a], Dae San Kim[b], Taekyun Kim[c], Toufik Mansour[d,*]

[a] Institute of Natural Sciences, Far Eastern Federal University, Vladivostok, 690950, Russia
[b] Department of Mathematics, Sogang University, Seoul 121-742, South Korea
[c] Department of Mathematics, Kwangwoon University, Seoul, South Korea
[d] Department of Mathematics, University of Haifa, Haifa 3498838, Israel

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ABSTRACT

In this paper, we consider the Barnes-type degenerate Euler polynomials. We present several explicit formulas and recurrence relations for these polynomials. Also, we establish a connection between our polynomials and several known families of polynomials.

1. Introduction

The goals of this paper are to use umbral calculus to obtain several new and interesting identities of Barnes-type degenerate Euler polynomials. Umbral calculus has been used in numerous problems of mathematics (for example, see [1,4,5,7,11–14,17]) and used in different areas of physics; for example it is used in group theory and quantum mechanics by Biedenharn et al. [2,3] (for other examples, see [12] and references therein).

Recall that the *Bernoulli numbers of order s* (see [9,10]) are defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)^s = \sum_{\ell\geq 0} B_\ell^{(s)}\frac{t^\ell}{\ell!}. \tag{1}$$

For $\mu \in \mathbb{C}$ with $\mu \neq 1$, the *Frobenius–Euler polynomials of order s* (see [6,7]) are defined by the generating function to be

$$\left(\frac{1-\mu}{e^t-\mu}\right)^s e^{tx} = \sum_{\ell\geq 0} H_\ell^{(s)}(x|\mu)\frac{t^\ell}{\ell!}. \tag{2}$$

For $r \in \mathbb{Z}_{>0}$, the *Barnes-type degenerate Euler polynomials* $E_n(\lambda, x|a_1, \ldots, a_r)$ are defined by the generating function

$$\prod_{i=1}^{r}\left(\frac{2}{(1+\lambda t)^{a_i/\lambda}+1}\right)(1+\lambda t)^{x/\lambda} = \sum_{n\geq 0} E_n(\lambda, x|a_1,\ldots,a_r)\frac{t^n}{n!}. \tag{3}$$

* Corresponding author. Tel.: +972 4 8240705; fax: +972 4 8240024.
E-mail addresses: d_dol@mail.ru (D.V. Dolgy), dskim@sogang.ac.kr (D.S. Kim), taekyun64@hotmail.com (T. Kim), toufik@math.haifa.ac.il, tmansour@univ.haifa.ac.il (T. Mansour).

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If $x = 0$, $E_n(\lambda|a_1, \ldots, a_r) = E_n(\lambda, 0|a_1, \ldots, a_r)$ are called the *Barnes-type degenerate Euler numbers*. We note here that

$$\lim_{\lambda\to 0} E_n(\lambda, x|a_1, \ldots, a_r) = E_n(x|a_1, \ldots, a_r),$$

$$\lim_{\lambda\to\infty} \lambda^{-n} E_n(\lambda, \lambda x|a_1, \ldots, a_r) = (x)_n = x(x-1)(x-2)\cdots(x-n+1),$$

where $\prod_{i=1}^{r}(\frac{2}{e^{a_i t}+1})e^{tx} = \sum_{n\geq 0} E_n(x|a_1, \ldots, a_r)\frac{t^n}{n!}$. For $x = r = a_1 = 1$, we write $E_n(1) = E_n(1|1)$ (see [9,10]). In the special case of $a_1 = \cdots = a_r = 1$, $E_n^{(r)}(\lambda, x) = E_n(\lambda, x|1, \ldots, 1)$ are called the *degenerate Euler polynomials of order r* (see [8]).

In order to study the Barnes-type degenerate Euler polynomials, we use of the umbral calculus technique. Let Π be the algebra of polynomials in a single variable x over $\mathbb{C}$. We denote the vector space of all linear functionals on Π by Π^*. The action of a linear functional $L \in \Pi^*$ on a polynomial $p(x)$ is denoted by $\langle L|p(x)\rangle$, and linearly extended as $\langle cL + c'L'|p(x)\rangle = c\langle L|p(x)\rangle + c'\langle L'|p(x)\rangle$, where $c, c' \in \mathbb{C}$ (see [15,16]). Let

$$\mathcal{H} = \left\{ f(t) = \sum_{k\geq 0} a_k \frac{t^k}{k!} \mid a_k \in \mathbb{C} \right\} \tag{4}$$

be the algebra of formal power series in a single variable t. The formal power series in the variable t defines a linear functional on Π by setting $\langle f(t)|x^n\rangle = a_n$, for all $n \geq 0$ (see [15,16]). So, by (4), we have

$$\langle t^k|x^n\rangle = n!\delta_{n,k}, \text{ for all } n, k \geq 0, (\text{see [15,16]}), \tag{5}$$

where $\delta_{n,k}$ is the Kronecker's symbol. Let $f_L(t) = \sum_{n\geq 0}\langle L|x^n\rangle \frac{t^n}{n!}$. By (5), we have that $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. So, the map $L \mapsto f_L(t)$ is a vector space isomorphism from Π^* onto $\mathcal{H}$. Thus, $\mathcal{H}$ is thought of as set of both formal power series and linear functionals. We call $\mathcal{H}$ the *umbral algebra*. The *umbral calculus* is the study of umbral algebra.

The *order* $O(f(t))$ of the non-zero power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish (see [15,16]). If $O(f(t)) = 1$ (respectively, $O(f(t)) = 0$), then $f(t)$ is called a *delta* (respectively, an *invertable*) series. Suppose that $O(f(t)) = 1$ and $O(g(t)) = 0$, then there exists a unique sequence $s_n(x)$ of polynomials such that $\langle g(t)(f(t))^k|s_n(x)\rangle = n!\delta_{n,k}$, where $n, k \geq 0$. The sequence $s_n(x)$ is called the *Sheffer sequence* for $(g(t), f(t))$, and we write $s_n(x) \sim (g(t), f(t))$ (see [15,16]). For $f(t) \in \mathcal{H}$ and $p(x) \in \Pi$, we have that $\langle e^{yt}|p(x)\rangle = p(y)$, $\langle f(t)g(t)|p(x)\rangle = \langle g(t)|f(t)p(x)\rangle$, $f(t) = \sum_{n\geq 0}\langle f(t)|x^n\rangle \frac{t^n}{n!}$ and $p(x) = \sum_{n\geq 0}\langle t^n|p(x)\rangle \frac{x^n}{n!}$. Therefore,

$$\langle t^k|p(x)\rangle = p^{(k)}(0), \quad \langle 1|p^{(k)}(x)\rangle = p^{(k)}(0), \tag{6}$$

where $p^{(k)}(0)$ denotes the kth derivative of $p(x)$ with respect to x at $x = 0$. So, by (6), we get that $t^k p(x) = p^{(k)}(x) = \frac{d^k}{dx^k}p(x)$, for all $k \geq 0$, (see [15,16]).

Let $s_n(x) \sim (g(t), f(t))$. Then we have

$$\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{n\geq 0} s_n(y)\frac{t^n}{n!}, \tag{7}$$

for all $y \in \mathbb{C}$, where $\bar{f}(t)$ is the compositional inverse of $f(t)$ (see [15,16]). For $s_n(x) \sim (g(t), f(t))$ and $r_n(x) \sim (h(t), \ell(t))$, let $s_n(x) = \sum_{k=0}^{n} c_{n,k} r_k(x)$, then we have

$$c_{n,k} = \frac{1}{k!}\left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))}(\ell(\bar{f}(t)))^k \middle| x^n \right\rangle, \tag{8}$$

(see [15,16]).

By the theory of Sheffer sequences, it is immediate that the Barnes-type degenerate Euler polynomial is the Sheffer sequence for the pair $g(t) = \prod_{i=1}^{r}\left(\frac{e^{a_i t}+1}{2}\right)$ and $f(t) = \frac{1}{\lambda}(e^{\lambda t} - 1)$. Thus

$$E_n(\lambda, x|a_1, \ldots, a_r) \sim \left(\prod_{i=1}^{r}\left(\frac{e^{a_i t}+1}{2}\right), \frac{1}{\lambda}(e^{\lambda t} - 1)\right). \tag{9}$$

The aim of the present paper is to present several new identities for Barnes-type degenerate Euler polynomials by the use of umbral calculus.

2. Explicit expressions

In this section we suggest several explicit formulas for the Barnes-type degenerate Euler polynomials. In order to do that, we recall that the Stirling numbers $S_1(n, m)$ of the first kind are defined as $(x)_n = \sum_{m=0}^{n} S_1(n, m)x^m \sim (1, e^t - 1)$ or $\frac{1}{j!}(\log(1+t))^j = \sum_{\ell\geq j} S_1(\ell, j)\frac{t^\ell}{\ell!}$. Define $(x|\lambda)_n$ to be $\lambda^n(x/\lambda)_n$, for all n. Also, we define

$$P_r(t) = \prod_{i=1}^{r}\left(\frac{2}{(1+\lambda t)^{a_i/\lambda} + 1}\right).$$