Applied Mathematics and Computation 266 (2015) 6–11

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A generalization of Jain's operators

A. Olgun [a], F. Taşdelen [b], A. Erençin [c,*]

[a] Kirikkale University, Faculty of Science and Arts, Department of Mathematics, 71450, Yahşihan, Kirikkale, Turkey
[b] Ankara University, Faculty of Science, Department of Mathematics, 06100, Tandoğan, Ankara, Turkey
[c] Abant İzzet Baysal University, Faculty of Science and Arts, Department of Mathematics, Bolu, 14280, Turkey

ARTICLE INFO

MSC:
41A25
41A36

Keywords:
Jain's operator
Weighted approximation
Voronovskaya type result

ABSTRACT

In this paper, we introduce a generalization of Jain's operators based on a function ρ. We examine convergence properties of such operators and prove a Voronovskaya type result.

1. Introduction

In [8], Jain proposed a new class of positive linear operators with the help of a Poisson type distribution as follows:

$$P_n^{[\beta]}(f;x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) w_\beta(k,nx), \quad x \geq 0, n \in \mathbb{N},$$

where

$$w_\beta(k,nx) = nx(nx+k\beta)^{k-1}\frac{e^{-(nx+k\beta)}}{k!}, \quad 0 \leq \beta < 1$$

and

$$\sum_{k=0}^{\infty} w_\beta(k,nx) = 1,$$

and studied convergence property and the order of approximation of these operators on a finite closed interval of $\mathbb{R}^+$ under the restriction $\beta \to 0$ as $n \to \infty$. Later, Farcaş [4] proved a Voronovskaya type theorem for Jain's operators. Recently, in [1], Agratini obtained local approximation results and statistical convergence property of the positive linear operators $P_n^{[\beta]}$. Note that when $\beta = 0$ the operators defined by Jain reduce to the well known Szász–Mirakyan operators

$$S_n(f;x) = e^{-nx}\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\frac{(nx)^k}{k!}, \quad x \geq 0, n \in \mathbb{N}.$$

* Corresponding author. Tel.: +90 3742541000.
E-mail addresses: aolgun@kku.edu.tr (A. Olgun), tasdelen@science.ankara.edu.tr (F. Taşdelen), erencina@hotmail.com (A. Erençin).

http://dx.doi.org/10.1016/j.amc.2015.05.060
0096-3003/

In 2014, Aral et al. [2] introduced a generalization of Szász–Mirakyan operators based on a function ρ as

$$\begin{aligned} S_n^{\rho}(f;x) &= e^{-n\rho(x)}\sum_{k=0}^{\infty}\left(f\circ\rho^{-1}\right)\left(\frac{k}{n}\right)\frac{(n\rho(x))^k}{k!} \\ &= \left(S_n\left(f\circ\rho^{-1}\right)\circ\rho\right)(x) \\ &= e^{-n\rho(x)}\sum_{k=0}^{\infty} f\left(\rho^{-1}\left(\frac{k}{n}\right)\right)\frac{(n\rho(x))^k}{k!}, \end{aligned}$$

where ρ is a function such that

(ρ_1) ρ is a continuously differentiable function on $\mathbb{R}^+$
(ρ_2) $\rho(0)=0$, $\inf_{x\in\mathbb{R}^+}\rho'(x)\geq 1$.

We observe from the above conditions we have $\lim_{x\to\infty}\rho(x)=\infty$ and $|t-x|\leq|\rho(t)-\rho(x)|$ for all $x,t\in\mathbb{R}^+$. The authors proved a weighted convergence theorem and a Voronovskaya type theorem and also obtained the degree of approximation by means of the weighted modulus of continuity for these operators. Moreover, they studied some preservation properties of $S_n^{\rho}(f;x)$.

Motivated with this work, we consider a generalization of the linear positive operators $P_n^{[\beta]}(f;x)$ as follows:

$$\begin{aligned} S_n^{\beta,\rho}(f;x) &= \sum_{k=0}^{\infty}\left(f\circ\rho^{-1}\right)\left(\frac{k}{n}\right)w_\beta(k,n\rho(x)) \\ &= \sum_{k=0}^{\infty} f\left(\rho^{-1}\left(\frac{k}{n}\right)\right)w_\beta(k,n\rho(x)),\quad x\geq 0, n\in\mathbb{N} \end{aligned} \tag{1.1}$$

where $w_\beta(k,n\rho(x))$ defined as in $P_n^{[\beta]}(f;x)$ and the function ρ has the properties given by (ρ_1) and (ρ_2). Note that for $\beta=0$ and additionally $\rho(x)=x$ the operators $S_n^{\beta,\rho}(f;x)$ turn out to be the operators $S_n^{\rho}(f;x)$ and $S_n(f;x)$, respectively.

In the present paper, we firstly introduce convergence property via weighted Korovkin type theorem given in [5,6] and find an estimate with the help of the weighted modulus of continuity defined in [7]. Furthermore, we prove a Voronovskaya type result for the operators $S_n^{\beta,\rho}(f;x)$.

2. Convergence of $S_n^{\beta,\rho}(f;x)$

Firstly, we introduce some auxiliary results.

Lemma 2.1. *For the operators defined by (1.1) we have*

$$S_n^{\beta,\rho}(1;x)=1 \tag{2.1}$$

$$S_n^{\beta,\rho}(\rho;x)=\frac{\rho(x)}{1-\beta} \tag{2.2}$$

$$S_n^{\beta,\rho}(\rho^2;x)=\frac{\rho^2(x)}{(1-\beta)^2}+\frac{\rho(x)}{n(1-\beta)^3} \tag{2.3}$$

$$S_n^{\beta,\rho}(\rho^3;x)=\frac{\rho^3(x)}{(1-\beta)^3}+\frac{3\rho^2(x)}{n(1-\beta)^4}+\frac{(2\beta+1)\rho(x)}{n^2(1-\beta)^5} \tag{2.4}$$

and

$$S_n^{\beta,\rho}(\rho^4;x)=\frac{\rho^4(x)}{(1-\beta)^4}+\frac{6\rho^3(x)}{n(1-\beta)^5}+\frac{(8\beta+7)\rho^2(x)}{n^2(1-\beta)^6}+\frac{(6\beta^2+8\beta+1)\rho(x)}{n^3(1-\beta)^7}. \tag{2.5}$$

Using the recurrence formula given in [8] it can be proved. So we omit it. Now from the linearity of the operators $S_n^{\beta,\rho}(f;x)$ we can state the following lemma.

Lemma 2.2. *For the operators defined by (1.1) we have*

$$S_n^{\beta,\rho}(\rho(t)-\rho(x);x)=\frac{\beta\rho(x)}{1-\beta} \tag{2.6}$$

$$S_n^{\beta,\rho}\left((\rho(t)-\rho(x))^2;x\right)=\frac{\beta^2\rho^2(x)}{(1-\beta)^2}+\frac{\rho(x)}{n(1-\beta)^3} \tag{2.7}$$