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A numerical investigation of time-fractional modified Fornberg–Whitham equation for analyzing the behavior of water waves

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ABSTRACT

In this paper, a new wavelet method based on the Hermite wavelet expansion together with operational matrices of fractional integration and derivative of wavelet functions is proposed to solve time-fractional modified Fornberg–Whitham (mFW) equation. The approximate solutions of time fractional modified Fornberg–Whitham equation which are obtained by Hermite wavelet method are compared with the exact solutions as well as the solutions obtained by optimal homotopy asymptotic method (OHAM). The present numerical scheme is quite simple, effective and expedient for obtaining numerical solution of fractional modified Fornberg–Whitham equation.

1. Introduction

The calculus of integrals and derivatives of arbitrary real or complex order (generally known as Fractional calculus) has gained widespread popularity during the past three decades or so, mostly due to its demonstrated applications in various fields of engineering and science.

The Fornberg–Whitham equation is given by [1]

$$\frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial x^2 \partial t} + \frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} = 3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + u\frac{\partial^3 u}{\partial x^3}, \tag{1.1}$$

which was first proposed by Whitham in 1967 for studying the qualitative behavior of wave breaking [2]. In 1978, Fornberg and Whitham [3] obtained a peaked solution consisting of an arbitrary constant. Modifying the nonlinear term $u\frac{\partial u}{\partial x}$ in (1.1) to $u^2\frac{\partial u}{\partial x}$, He et al. proposed in [1] the modified Fornberg–Whitham equation as follows

$$\frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial x^2 \partial t} + \frac{\partial u}{\partial x} + u^2\frac{\partial u}{\partial x} = 3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + u\frac{\partial^3 u}{\partial x^3}, \quad t > 0, \quad x > 0 \tag{1.2}$$

Consider the following time-fractional modified Fornberg–Whitham equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^3 u}{\partial x^2 \partial t} + \frac{\partial u}{\partial x} + u^2\frac{\partial u}{\partial x} = 3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + u\frac{\partial^3 u}{\partial x^3}, \tag{1.3}$$

Here $0 < \alpha \leq 1$, is the parameter representing the order of the fractional time derivative. The fractional derivative is considered in the Caputo sense.

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A great deal of research work has been invested in recent years for the study of classical order modified Fornberg–Whitham equations. Various methods such as the bifurcation theory and the method of phase portraits analysis [1], reduced differential transform method [4], and variational iteration method [5] have been developed independently for the solution of modified Fornberg–Whitham equation. But according to the best possible information of the authors, the detailed study of the nonlinear fractional order modified Fornberg–Whitham equation is only beginning. Our aim in the present work is to implement two-dimensional Hermite wavelet and optimal homotopy asymptotic method in order to exhibit the capabilities of these methods in handling nonlinear equation like fractional order modified Fornberg–Whitham equation.

This paper is devoted to study the time-fractional modified Fornberg–Whitham equation. Therefore, the present paper emphasizes on the application of two-dimensional Hermite wavelet technique for solving the problem of time-fractional modified Fornberg–Whitham equation. The obtained numerical approximate results of this method are then compared with the exact solutions and also with OHAM solutions for fractional order.

The framework of the paper is as follows: in Section 2, a quick review of the theory of fractional calculus has been provided for precise purpose of this paper. The mathematical preliminaries of Hermite wavelet is presented in Section 3. The approximation of function using Hermite wavelet and the operational matrices are presented in Sections 4 and 5 respectively. The basic idea of optimal homotopy asymptotic method (OHAM) is discussed in Section 6. To determine the approximate solution for the nonlinear time-fractional modified Fornberg–Whitham equation, two-dimensional Hermite wavelet method has been applied in Section 7. In Section 8, algorithm of Hermite wavelet has been presented. In Section 9, comparison of proposed method with regard to OHAM for the solution of time-fractional modified Fornberg–Whitham equation has been discussed. The convergence of two-dimensional Hermite wavelet is discussed in Section 10. The numerical results and discussions are discussed in Section 11 and Section 12 concludes the paper.

2. Fractional derivative and integration

Definition. The Riemann–Liouville fractional integral operator J^α $(\alpha > 0)$ of a function $f(t)$,is defined as [6,7]

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \quad \alpha > 0 \text{ and } \alpha \in \Re^+ \tag{2.1}$$

Some of the properties of the operator J^α are as follows:

$$J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t), \quad (\alpha > 0, \beta > 0) \tag{2.2}$$

$$J^\alpha t^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\alpha)} t^{\alpha+\gamma}, \quad (\gamma > -1) \tag{2.3}$$

Definition. The Caputo fractional derivative ${}_0D_t^\alpha$ of a function $f(t)$ is defined as [6,7]

$${}_0D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad (n-1 < \alpha \le n, n \in N) \tag{2.4}$$

Some properties of the Caputo fractional derivative are as follows:

$${}_0D_t^\alpha t^\beta = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha}, \quad 0 < \alpha < \beta+1, \quad \beta > -1 \tag{2.5}$$

$$J^\alpha D^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}\left(0^+\right)\frac{t^k}{k!} \quad n-1 < \alpha \le n \text{ and } n \in N \tag{2.6}$$

3. Hermite wavelets

The Hermite polynomials $H_m(x)$ of order m are defined on the interval $(-\infty, \infty)$, and can be determined with the aid of the following recurrence formulae:

$$H_0(x) = 1,$$

$$H_1(x) = 2x,$$

$$H_{m+1}(x) = 2xH_m(x) - 2mH_{m-1}(x), \quad m = 1, 2, 3, \ldots$$

The Hermite polynomials $H_m(x)$ are orthogonal with respect to the weight function e^{-x^2}.

The Hermite wavelets are defined on interval [0, 1) by [8]

$$\psi_{n,m}(x) = \begin{cases} 2^{k/2}\sqrt{\dfrac{1}{n!2^n\sqrt{\pi}}}H_m\left(2^k x - \hat{n}\right), & \text{for } \dfrac{\hat{n}-1}{2^k} \le x < \dfrac{\hat{n}+1}{2^k} \\ 0, & \text{otherwise} \end{cases} \tag{3.1}$$