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Monotonicity of zeros for a class of polynomials including hypergeometric polynomials

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ABSTRACT

We study the monotonicity of zeros in connection with perturbed recurrence coefficients of polynomials satisfying certain three-term recurrence relations of Frobenius-type. These recurrence relations are the key ingredient for the tridiagonal approach developed by Delsarte and Genin to solve the standard linear prediction problem. As a particular case, we consider the Askey para-orthogonal polynomials on the unit circle, ${}_2F_1(-n, a+bi; 2a; 1-z)$, $a, b \in \mathbb{R}$, extending a recent result about the monotonicity of their zeros with respect to the parameter b. Finally, the consequences of our results in the theory of orthogonal polynomials on the real line are discussed.

1. Introduction and main results

A sequence of real numbers $\{\alpha_n\}_{n\geq 1}$ is said to be a *chain sequence* if there exists a parameter sequence $\{\chi_n\}_{n\geq 0}$ such that

$$0 \leq \chi_0 < 1, \quad 0 < \chi_n < 1, \quad \alpha_n := (1-\chi_{n-1})\chi_n, \quad n \geq 1.$$

In a more general setting this definition was introduced by Wall [45] in connection with continued fractions. A parameter sequence $\{m_n\}_{n\geq 0}$ associated with the chain sequence $\{\alpha_n\}_{n\geq 1}$ is called its *minimal parameter sequence* if $m_0 = 0$. We recommend the reader to consult the classical monographs by Wall [45] and Chihara [14] containing a discussion of these and related topics.

The explicit relation between chain sequences and *orthogonal polynomials on the real line* (OPRL, in short) was founded by Chihara in [13], and later studied by Chihara himself [14], and several authors including Lasser [30], Van Doorn [20], and Szwarc [42,43], among others. A recent appearance of chain sequences in the theory of *orthogonal polynomials on the unit circle* (OPUC, in short), also known as Szegő's polynomials, can be found in [9] and references therein.

In [9] it was obtained a characterization of the OPUC in terms of the minimal parameter sequence $\{m_n\}_{n\geq 0}$ and a sequence of polynomials $\{s_n\}_{n\geq 0}$ that satisfies the three-term recurrence relation

$$s_{n+1}(z) = ((1+i\beta_{n+1})z + (1-i\beta_{n+1}))s_n(z) - 4\alpha_n z s_{n-1}(z), \tag{1.1}$$

with initial conditions $s_0 := 1$ and $s_1(z) := (1+i\beta_1)z + (1-i\beta_1)$. Here, $\{\beta_n\}_{n\geq 1}$ is an arbitrary real sequence. If we set $s_n(z) := 0$ for $n < 0$ and $\alpha_n := \beta_n := 0$ for $n < 1$, then (1.1) holds for every $n \in \mathbb{Z}$. In [9], among others, the authors have shown that associated with the chain sequence $\{\alpha_n\}_{n\geq 1}$ and the real sequence $\{\beta_n\}_{n\geq 1}$ there exists a unique non-trivial probability measure supported

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on $\partial\mathbb{D} := \{z \in \mathbb{C};\ |z| = 1\}$, such that

$$\Phi_0(z) := 1, \quad \Phi_n(z)\prod_{k=1}^{n}(1 + i\beta_k) := s_n(z) - 2(1 - m_n)s_{n-1}(z), \tag{1.2}$$

is the corresponding sequence of monic OPUC.

It can be noted that (1.1) is equivalent to the three-term recurrence relation of Frobenius-type [23] studied by Delsarte and Genin [16–18]. The weakened form of orthogonality, known as para-orthogonality after [29], that the polynomials $\{s_n\}_{n\geq 0}$ satisfy with respect to the orthogonality measure of the associated OPUC was also pointed out in [16]. In other words, these polynomials are *para-orthogonal polynomials on the unit circle* (POPUC, in short) [29]. We recall that (1.1) is the key ingredient for the *tridiagonal approach* developed by Delsarte and Genin to solve the standard linear prediction problem [16–18]. Unfortunately, these papers must have avoided the attention of some people working in POPUC. It is interesting to compare (1.2) with [16, Eq. 3.1]. Notice that in [16, Section 3], the authors studied the one-to-one correspondence between the sequence $\{s_n\}_{n\geq 0}$ and $\{\Phi_n\}_{n\geq 0}$.

It is well known [7,16,24,36–38,47] that the zeros of the polynomials $\{s_n\}_{n\geq 0}$ are all simple and located on $\partial\mathbb{D}$. Moreover, the zeros of two consecutive polynomials strictly interlace on $\partial\mathbb{D}$ [16, Section 5], although in [9,19] the previous result was proved using the ideas contained in [15, Section IV] and [16, Section 2], among others. More specifically, if one denotes the zeros of the polynomial of degree n, $s_n(z)$, by $z_{n,j} = e^{i\theta_{n,j}}$, $j = 1, 2, \ldots, n$, then

$$0 < \theta_{n,1} < \theta_{n-1,1} < \theta_{n,2} < \cdots < \theta_{n,n-1} < \theta_{n-1,n-1} < \theta_{n,n} < 2\pi. \tag{1.3}$$

For this, an auxiliary function was introduced

$$f_n(x) := (4z)^{-n/2}s_n(z),$$

where

$$2x := z^{1/2} + z^{-1/2}, \quad z = e^{i\theta}, \quad \theta \in [0, 2\pi], \tag{1.4}$$

and $(re^{i\theta})^{1/2} = \sqrt{r}e^{i\theta/2}$, $r > 0$. The correspondence between the complex variable $z = e^{i\theta}$, $\theta \in [0, 2\pi]$, and the real variable $x \in [-1, 1]$ given by (1.4) is one-to-one and it is called the Delsarte–Genin mapping [15,48] with scaling parameter 1/2. This mapping in an implicit form is contained in [45, Chapter 15].

From the previous mapping, the function $f_n(x)$ has exactly the same number of zeros in $[-1, 1]$ that $s_n(z)$ on $\partial\mathbb{D}$. Furthermore, it is easily seen that the sequence of functions $\{f_n\}_{n\geq 0}$ satisfies the three-term recurrence relation

$$f_n(x) = (x - \beta_n\sqrt{1 - x^2})f_{n-1}(x) - \alpha_{n-1}f_{n-2}(x), \quad n \geq 2, \tag{1.5}$$

with initial conditions $f_0(x) = 1$ and $f_1(x) = x - \beta_1\sqrt{1 - x^2}$, see also [16, Eq. 2.7]. By an inductive argument, it was established [19] that if one denotes the zeros of $f_n(x)$ by $x_{n,j} = \cos(\theta_{n,j}/2)$, $j = 1, 2, \ldots, n$, then

$$-1 < x_{n,n} < x_{n-1,n-1} < x_{n,n-1} < \cdots < x_{n,2} < x_{n-1,1} < x_{n,1} < 1. \tag{1.6}$$

Thus, (1.3) holds. We present this approach because we will use (1.5) to generalize the original result obtained in [19], although the natural way is deduce (1.6) from the known result (1.3). Notice also that $\{f_n\}_{n\geq 0}$ is a sequence of symmetric orthogonal polynomials on the interval $[-1, 1]$ when $\beta_n = 0$ for every $n\geq 1$ [18, Section 5].

For complex numbers a, b, and c, where $c \neq 0, -1, -2, \ldots$, the Gaussian hypergeometric function ${}_2F_1(a, b; c; z)$ is defined by the series

$$\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$$

for $|z| < 1$, and by analytic continuation elsewhere, see [1] and the references given there. Here $(a)_n$ is the Pochhammer symbol given by $(a)_0 := 1$ and $(a)_n := a(a+1)\cdots(a+n-1)$, $n\geq 1$.

In [19], the monotonicity of the zeros, using a slightly modified Sturm comparison theorem (see also [32]), for a special class of polynomials satisfying (1.1) was also studied, the POPUC ${}_2F_1(-n, a+bi; 2a; 1-z)$, $a, b \in \mathbb{R}$. The results obtained therein can be extended to (1.1). As a matter of fact the results obtained from (1.1) are more interesting from the point of view of the works of Delsarte and Genin previously cited. In the present work we give the complete solution of the corresponding problem about the monotonicity of the zeros of the polynomials $\{s_n\}_{n\geq 0}$ with respect to a parameter that appears in the sequence of real recurrence coefficients $\{\beta_n\}_{n\geq 1}$.

Our main results read as follows.

Theorem A. *Let $\{s_n(\epsilon;\cdot)\}_{n\geq 0}$ be a sequence of polynomials given by (1.1) with perturbed recurrence coefficients depending on a real parameter $\epsilon \neq 0$ such that*

$$\begin{aligned}\beta_{n+1}(\epsilon) &= \beta_{n+1} + \epsilon\delta_{n+1,k},\\ \alpha_{n+1}(\epsilon) &= \alpha_{n+1},\end{aligned} \tag{1.7}$$