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On the existence of solutions for fractional differential inclusions with sum and integral boundary conditions

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ABSTRACT

In this paper, by using the endpoint result for multifunctions, we investigate the existence of solutions for a boundary value problem for fractional differential inclusions with sum and integral boundary conditions. Finally, an example is also given to illustrate the validity of our main result.

1. Introduction and preliminaries

The topic of fractional differential equations is one of the branches of mathematics which has various important applications in many fields as mathematics, physics, chemistry, biology and many other branches of engineering. Many papers have published about fractional differential equations by researchers which apply the fixed point theory in their existence theorems. For instence, one can find a lot of papers in this field (see [1–5,10–15,17], and the references therein). Also, some researchers have been investigated the existence of solutions for some fractional differential inclusions (see for example, [2,3,6–8,16,25,28,30]).

Let $\alpha > 0$, $n-1 < \alpha < n$, $n = [\alpha]+1$ and $u \in C([a,b],\mathbb{R})$. The Caputo derivative of fractional of order α for the function u is defined by ${}^cD_0^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}u^{(n)}(\tau)d\tau$ (see for more details [20–22,27,29,31]). Also, the Riemann–Liouville fractional order integral of the function u is defined by $I_0^\alpha u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{u(\tau)}{(t-\tau)^{1-\alpha}}d\tau$ $(t>0)$ whenever the integral exists ([20–22,27,29,31]). In [26], it has been proved that the general solution of the fractional differential equation ${}^cD_0^\alpha u(t) = 0$ is given by $u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1}$, where $c_0, \ldots, c_{n-1}$ are real constants and $n = [\alpha]+1$. Also, for each $T > 0$ and $u \in C([0,T])$ we have

$$I_0^\alpha \, {}^cD_0^\alpha u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1},$$

where $c_0, \ldots, c_{n-1}$ are real constants and $n = [\alpha]+1$ ([26]).

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Now, we review some definitions and notations about multifunctions. Let (X, d) be a metric space. Denote by $\mathcal{P}(X)$ the class of all nonempty subsets of X. Hence, $\mathcal{P}_{cl}(X)$, $\mathcal{P}_{bd}(X)$, $\mathcal{P}_{cv}(X)$ and $\mathcal{P}_{cp}(X)$ denote the class of all closed, bounded, convex and compact subsets of X, respectively. A mapping $F : X \to \mathcal{P}(X)$ is called a multifunction on X and $u \in X$ is called a fixed point of F whenever $u \in Fu$ ([18,19,23]). The Pompeiu–Hausdorff metric $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to [0, \infty)$ is defined by $H_d(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\}$, where $d(A, b) = \inf_{a\in A} d(a, b)$ ([19]). Then $(\mathcal{P}_{cl,bd}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space, where $\mathcal{P}_{cl,bd}(X)$ is the set of closed and bounded subsets of X. ([19]).

A multifunction $F : J \to \mathcal{P}_{cl}(\mathbb{R})$ is said to be measurable if the function $t \mapsto d(y, F(t))$ is measurable for all $y \in \mathbb{R}$, where $J = [0, 1]$ ([19]). A multifunction $F : X \to \mathcal{P}_{cl}(X)$ is called a contraction if there exists $\gamma \in (0, 1)$ such that $H_d(F(x), F(y)) \le \gamma d(x, y)$ for all $x, y \in X$ ([19]).

In this paper, we study the following Caputo fractional differential inclusion

$$ {}^cD_0^{\alpha} u(t) \in F(t, u(t), u'(t), {}^cD_0^{p_1} u(t), \dots, {}^cD_0^{p_k} u(t)), \tag{1.1}$$

supplement with sum and integral boundary conditions

$$u(0) + \sum_{j=1}^{m} b_j u''(0) = 0, \quad \gamma_1 u(\eta) + \gamma_2 \int_0^1 u(\tau)d\tau = 0, \quad \sum_{j=1}^{m} b_j u'(1) + \gamma_3 \int_0^1 u(\tau)d\tau = 0, \tag{1.2}$$

where ${}^cD_0^{\alpha}$ denote the Caputo fractional derivative of order α, $t \in [0, 1] := J$, $2 < \alpha \le 3$, $1 < p_i \le 2$, $(i = 1, \dots, k;\ k \ge 1)$, $0 < \eta < 1$, $b_j (j = 1, \dots, m;\ m \ge 1)$, $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$ and $F : J \times \mathbb{R}^{k+2} \to \mathcal{P}(\mathbb{R})$ is a compact-valued multifunction.

An existence result for the boundary value problems (1.1)and (1.2) is proved, via endpoint theory. An element $x \in X$ is called *an endpoint of a multifunction* $F : X \to \mathcal{P}(X)$ whenever $Fx = \{x\}$ ([9]). Also, we say that F *has an approximate endpoint property* whenever $\inf_{x\in X} \sup_{y\in Fx} d(x, y) = 0$ ([9]). A function $f : \mathbb{R} \to \mathbb{R}$ is called upper semi-continuous whenever $\limsup_{n\to\infty} f(\lambda_n) \le f(\lambda)$ for all sequence $\{\lambda_n\}_{n\ge 1}$ with $\lambda_n \to \lambda$.

For the proof of our main result we use the following endpoint fixed point theorem of Amini–Harandi [9].

Lemma 1.1 ([9]). *Let $\psi: [0, \infty) \to [0, \infty)$ be an upper semi-continuous function such that $\psi(t) < t$ and $\liminf_{t\to\infty}(t - \psi(t)) > 0$ for all $t > 0$, (X, d) a complete metric space and $T : X \to CB(X)$ a multifunction such that $H_d(Tx, Ty) \le \psi(d(x, y))$ for all $x, y \in X$. Then T has a unique endpoint if and only if T has approximate endpoint property.*

2. Main result

Now, we are ready to prove our main result. Let $X = \{u : u, u', {}^cD_0^{p_i} u \in C(J, \mathbb{R})\}$ endowed with the norm $\|u\| = \sup_{t\in J} |u(t)| + \sup_{t\in J} |u'(t)| + \sum_{i=1}^{k} \sup_{t\in J} |{}^cD_0^{p_i} u(t)|$. Then, $(X, \|\cdot\|)$ is a Banach space [32].

Lemma 2.1. *Given $y \in X$, then the unique solution of the problem*

$$\begin{cases} {}^cD_0^{\alpha} u(t) = y(t), \quad 0 < t < 1, \quad 2 < \alpha \le 3, \\ u(0) + \sum_{j=1}^{m} b_j u''(0) = 0, \ \gamma_1 u(\eta) + \gamma_2 \int_0^1 u(\tau)d\tau = 0, \\ \sum_{j=1}^{m} b_j u'(1) + \gamma_3 \int_0^1 u(\tau)d\tau = 0, \end{cases} \tag{2.1}$$

is given by

$$\begin{aligned} u(t) = &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + \frac{\gamma_1}{\Delta} A(t) \int_0^{\eta} \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds \\ &+ \frac{1}{\Delta}(\gamma_2 A(t) + \gamma_3 B(t)) \int_0^1 \int_0^{\tau} \frac{(\tau-\omega)^{\alpha-1}}{\Gamma(\alpha)} y(\omega)d\omega d\tau \\ &+ \frac{1}{\Delta} B(t) \sum_{j=1}^{m} b_j \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds, \end{aligned} \tag{2.2}$$

where

$$A(t) = \left(\sum_{j=1}^{m} b_j + \frac{\gamma_3}{2}\right)\left(-2\sum_{j=1}^{m} b_j - t + t^2\right) + 2\gamma_3 t \sum_{j=1}^{m} b_j, \tag{2.3}$$

$$B(t) = \left(\gamma_1 \eta + \frac{\gamma_2}{2}\right)\left(2\sum_{j=1}^{m} b_j - t^2)\right) + t\left(\gamma_1 \eta^2 + \frac{\gamma_2}{3} - 2(\gamma_1 + \gamma_2)\sum_{j=1}^{m} b_j\right), \tag{2.4}$$