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Jensen–Ostrowski type inequalities and applications for f-divergence measures

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ABSTRACT

In this paper, we provide inequalities of Jensen–Ostrowski type, by providing bounds for the magnitude of

$$\int_\Omega (f\circ g)\,d\mu - f(\zeta) - \int_\Omega (g-\zeta) f'\circ g\,d\mu + \frac{1}{2}\lambda \int_\Omega (g-\zeta)^2\,d\mu,$$

for various assumptions on the absolutely continuous function $f : [a, b] \to \mathbb{C}$, $\zeta \in [a, b]$, $\lambda \in \mathbb{C}$, and a μ-measurable function g on Ω. Special cases are considered to provide some inequalities of Jensen type, as well as Ostrowski type, in measure-theoretic (probabilistic) form. Applications for f-divergence measure in information theory are also considered.

1. Introduction

In 1905 (1906) Jensen defined convex functions as follows: f is convex if

$$f\left(\frac{a+b}{2}\right) \le \frac{f(a)+f(b)}{2} \tag{1}$$

for all a and b in $D(f)$ (here $D(f)$ is the domain of f) [13]. Inequality (1) is the simplest form of Jensen's inequality. Jensen's inequality has been widely applied in many areas of research, e.g. probability theory, statistical physics, and information theory.

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space such that $\int_\Omega d\mu = 1$, consisting of a set Ω, a σ-algebra $\mathcal{A}$ of subsets of Ω, and a countably additive and positive measure μ on $\mathcal{A}$ with values in the set of extended real numbers. Consider the Lebesgue space

$$L(\Omega,\mu) := \left\{ f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(t)|d\mu(t) < \infty \right\}.$$

For simplicity of notation, we write in the text $\int_\Omega f d\mu$ instead of $\int_\Omega f(t)d\mu(t)$. Jensen's inequality now takes the following form: for a μ-integrable function $g : \Omega \to [m, M] \subset \mathbb{R}$, and a convex function $f : [m, M] \to \mathbb{R}$, we have

$$f\left(\int_\Omega g\,d\mu\right) \le \int_\Omega f\circ g\,d\mu. \tag{2}$$

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We refer the readers to [4] and [5] for the reverses of Jensen's inequality (2) (the discrepancy in Jensen's inequality) and their applications to divergence measures.

In 1938, Ostrowski [12] proved an inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b f(t)dt$ and the value $f(x)$ $(x \in [a, b])$:

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) such that $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b), i.e. $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \le \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right]\|f'\|_\infty (b-a), \tag{3}$$

for all $x \in [a, b]$ and the constant $\frac{1}{4}$ is the best possible.

Milovanović and Pečarić proved a generalisation of Ostrowski's inequality for n-time differentiable mappings [10] (cf. Mitrinović et. al [11]). The case of twice differentiable mappings is mentioned in Theorem 1.3 of [1], as follows:

Theorem 2. *Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable mapping such that $f'' : (a, b) \to \mathbb{R}$ is bounded on (a, b), that is, $\|f''\|_\infty = \sup_{t\in[a,b]} |f''(t)| < \infty$. Then, we have the following inequality for all $x \in (a, b)$:*

$$\begin{aligned}&\left|\frac{1}{2}\left[f(x) + \frac{(x-a)f(a) + (b-x)f(b)}{b-a}\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\quad\le \frac{\|f''\|_\infty}{4}(b-a)^2\left[\frac{1}{12} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right].\end{aligned} \tag{4}$$

Dragomir [7] introduced some inequalities which combine the two aforementioned inequalities, referred to as the Jensen–Ostrowski inequalities. These inequalities are established by obtaining bounds for the magnitude of

$$\int_\Omega f\circ g\,d\mu - f(\zeta) - \lambda\left(\int_\Omega g\,d\mu - \zeta\right), \quad \zeta \in [a, b],$$

for various assumptions on the absolutely continuous function $f : [a, b] \to \mathbb{C}$, a μ-measurable function g, and $\lambda \in \mathbb{C}$. In the same spirit, we investigate in this paper, the magnitude of

$$\int_\Omega (f\circ g)\,d\mu - f(\zeta) - \int_\Omega (g-\zeta)f'\circ g\,d\mu + \frac{1}{2}\lambda\int_\Omega (g-\zeta)^2\,d\mu, \quad \zeta \in [a, b],$$

to provide further inequalities of Jensen–Ostrowski type. We obtain some generalisations of Jensen's and Ostrowski's inequalities by setting $\zeta = \int_\Omega g\,d\mu$ and $\lambda = 0$, respectively. In particular, we provide a generalised version of inequality (4) in the measure-theoretic (and probabilistic) form. The paper is organised as follows. We provide some identities in Section 2 to assist us in the proofs of the main results. Inequalities with bounds involving the p-norms $(1 \le p \le \infty)$ are given in Section 3. Inequalities for functions with bounded second derivatives and convex second derivatives are given in Sections 4 and 5, respectively. Finally, an application for f-divergence measure in information theory are provided in Section 6.

2. Some identities

In this section, we give some identities which we use to assist us in proving the main results in Sections 3–5. Throughout the text, we denote by $\mathring{I}$, the interior of the set I.

Lemma 3. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$, $f' : [a, b] \subset \mathring{I} \to \mathbb{C}$ is absolutely continuous on $[a, b]$, and $\zeta \in [a, b]$. If $g : \Omega \to [a, b]$ is Lebesgue μ-measurable on Ω such that $f\circ g,\ (g-\zeta)f'\circ g,\ (g-\zeta)^2 \in L(\Omega, \mu)$, with $\int_\Omega d\mu = 1$, then for $\lambda \in \mathbb{C}$, we have*

$$\begin{aligned}&f(\zeta) + \int_\Omega (g-\zeta)f'\circ g\,d\mu - \frac{1}{2}\lambda\int_\Omega (g-\zeta)^2\,d\mu - \int_\Omega (f\circ g)\,d\mu \\ &\quad= \int_\Omega\left[(g-\zeta)^2\int_0^1 s\left[f''\big((1-s)\zeta + sg\big) - \lambda\right]ds\right]d\mu \qquad (5)\\ &\quad= \int_0^1 s\left[\int_\Omega (g-\zeta)^2\left[f''\big((1-s)\zeta + sg\big) - \lambda\right]d\mu\right]ds. \qquad (6)\end{aligned}$$

Proof. Since f is differentiable on $[a, b]$, hence for any u and v, we have

$$f(u) - f(v) = (u-v)\int_0^1 f'\big((1-s)v + su\big)\,ds. \tag{7}$$