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New efficient multipoint iterative methods for solving nonlinear systems

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ABSTRACT

It is attempted to put forward a new multipoint iterative method for approximating solutions of nonlinear systems. The main feature of the extended methods is that it uses only one LU factorization which preserves and reduces computational complexities. Moreover, the first step is designed in such a way that in most cases singularity of the denominator is avoided. Therefore, we try to generalize the suggested method so that we can increase the order of convergence from four to six and eight, but we do not need any new LU factorization. Also, we justify this advantage of the convergence analysis versus some numerical methods with different examples.

1. Introduction

Without doubt, solving nonlinear equations and more precisely nonlinear systems of equations has involved great attention of many researchers in science and engineering during the history of mathematics. Although iterative methods, single and multiple methods, for solving single-variable of nonlinear equations have been developed and studied very well recently [1,10], however, these methods cannot be easily applied or extended for solving of nonlinear systems.

There are two general ways for pursuing this aim analytically. The first one is based on the well-known n-dimensional Taylor expansion [6] and the second is based on the matrix approach, which is so-called Point of Attraction, introduced first in [2]. We here apply the first case for the sake of simplicity. This matter has been discussed in [3,4,6,8] thoroughly and one can consult them.

Let the function $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ has at least second-order Fréchet derivatives with continuity on an open set D. Suppose that the equation $F(x) = 0$ has a solution $x^* \in D$, that is $F(x^*) = 0$, where $F(x) = (f_1(x), f_2(x), \ldots, f_n(x))^T$, $x = (x_1, x_2, \ldots, x_n)^T$, and $f_i(x)$, $i = 1, 2, \ldots, n$, are real-valued functions.

Let F be sufficiently Fréchet differentiable in D. By using the notation introduced in [6], the qth derivative of F at $u \in \mathbb{R}^n$, $q \geq 1$, is the q-linear function $F^{(q)}(u) : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $F^{(q)}(u)(v_1, \ldots, v_q) \in \mathbb{R}^n$. It is well known that, for $x^* + h \in \mathbb{R}^n$ lying in a neighbourhood of a solution x^* of the nonlinear system $F(x) = 0$, Taylor's expansion can be applied and we have

$$F(x^* + h) = F'(x^*)\left[h + \sum_{q=2}^{p-1} C_q h^q\right] + \mathcal{O}(h^p), \tag{1.1}$$

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where $C_q = (1/q!)[F'(x^*)]^{-1}F^{(q)}(x^*),\ q \geq 2$. We observe that $C_q h^q \in \mathbb{R}^n$ since $F^{(q)}(x^*) \in \mathcal{L}(\mathbb{R}^n \times \cdots \times \mathbb{R}^n, \mathbb{R}^n)$ and $[F'(x^*)]^{-1} \in \mathcal{L}(\mathbb{R}^n)$. In addition, we can express F' as

$$F'(x^* + h) = F'(x^*)\left[I + \sum_{q=2}^{p-1} qC_q h^{q-1}\right] + \mathcal{O}(h^p), \tag{1.2}$$

wherein I is the identity matrix, and $qC_q h^{q-1} \in \mathcal{L}(\mathbb{R}^n)$.

It is widely known that Newton's method in several variables [2] could be written as

$$x^{(k+1)} = x^{(k)} - F'(x^{(k)})^{-1}F(x^{(k)}), \quad k = 0, 1, 2, \ldots, \tag{1.3}$$

where $x^{(0)}$ is the initial estimate and $F'(x^{(k)})$ is the Jacobian matrix of the function F evaluated in the kth iteration. This method has order of convergence two under certain conditions.

Another famous scheme for solving nonlinear systems of equations is Jarratt's fourth-order method [6] which is the generalisation of the scheme in the scalar case given in [5] as follows:

$$\begin{cases} y^{(k)} = x^{(k)} - \frac{2}{3}F'(x^{(k)})^{-1}F(x^{(k)}), \\ x^{(k+1)} = x^{(k)} - \frac{1}{2}[(3F'(y^{(k)}) - F'(x^{(k)}))^{-1}(3F'(y^{(k)}) + F'(x^{(k)}))]F'(x^{(k)})^{-1}F(x^{(k)}). \end{cases} \tag{1.4}$$

Sharma et al. [7] constructed, composed of two weighted Newton's step, the following fourth-order method

$$\begin{cases} y^{(k)} = x^{(k)} - \frac{2}{3}F'(x^{(k)})^{-1}F(x^{(k)}), \\ x^{(k+1)} = x^{(k)} - \frac{1}{2}\left[-I + \frac{9}{4}F'(y^{(k)})^{-1}F'(x^{(k)}) + \frac{3}{4}F'(x^{(k)})^{-1}F'(y^{(k)})\right]F'(x^{(k)})^{-1}F(x^{(k)}). \end{cases} \tag{1.5}$$

Another fourth-order Jarratt-type method has been devised by Babajee et al. [3] as following:

$$\begin{cases} y^{(k)} = x^{(k)} - \frac{2}{3}F'(x^{(k)})^{-1}F(x^{(k)}), \\ x^{(k+1)} = x^{(k)} - 2\left[I - \frac{1}{4}(F'(x^{(k)})^{-1}F'(y^{(k)}) - I) + \frac{3}{4}(F'(x^{(k)})^{-1}F'(y^{(k)}) - I)^2\right](F'(x^{(k)}) + F'(y^{(k)}))^{-1}F(x^{(k)}). \end{cases} \tag{1.6}$$

In the present study, we introduce three kinds of iterative methods for solving a system of nonlinear equations. For this purpose, first we suggest a new scheme and efficient two steps method which applies only one frozen LU decomposition. Moreover, the first step of this method contains a suitable factor in the denominator so that in almost all cases it avoids singularities. These are the main contributions and merits of the proposed method. To increase the convergence order, because of preserving LU factorization property, we can develop higher order of iterative methods based on the mentioned method. Consequently, we construct two new methods having convergence order of six and eight, respectively. Convergence analysis of these methods is proved. Numerical examples and comparisons will be illustrated to show the efficiency and applicability of our methods.

The paper is organized as follows: Section 2 is used to introduce two new methods for solving nonlinear systems of equations with four- and six-order convergence. In Section 3, our primary goal is to develop the four-step extended method. Indeed, we derive the eighth-order method. Numerical test problems and comparisons are illustrated in Section 4. The last section includes some conclusions.

2. Constructing iterative methods

In this section, we deal with construction of the new methods with convergence order of four and six, respectively. To this end, we consider a new method with fourth-order for solving nonlinear systems of equations as follows:

$$\begin{cases} y^{(k)} = x^{(k)} - (DF(x^{(k)}) - F'(x^{(k)}))^{-1}F(x^{(k)}), \\ x^{(k+1)} = y^{(k)} - \left(\frac{-3}{2}I + \frac{1}{2}(DF(x^{(k)}) - F'(x^{(k)}))^{-1}F'(y^{(k)})\right)((DF(x^{(k)}) - F'(x^{(k)}))^{-1}F(y^{(k)})), \end{cases} \tag{2.1}$$

where $DF(x)$ is a matrix that the elements of the vector $F(x)$ are on its diagonal and other elements of the matrix are zero. As we pointed it out before, the factor $DF(x)$ is crucial since it avoid singularities in almost all cases. This kind of modification has not been considered yet and it was out of motivation here. As a result, we can add another step to (2.1) and obtain the method with sixth-order as follows:

$$\begin{cases} y^{(k)} = x^{(k)} - (DF(x^{(k)}) - F'(x^{(k)}))^{-1}F(x^{(k)}), \\ z^{(k)} = y^{(k)} - \left(\frac{-3}{2}I + \frac{1}{2}(DF(x^{(k)}) - F'(x^{(k)}))^{-1}F'(y^{(k)})\right)((DF(x^{(k)}) - F'(x^{(k)}))^{-1}F(y^{(k)})), \\ x^{(k+1)} = z^{(k)} - \left(\frac{-3}{2}I + \frac{1}{2}(DF(x^{(k)}) - F'(x^{(k)}))^{-1}F'(y^{(k)})\right)((DF(x^{(k)}) - F'(x^{(k)}))^{-1}F(z^{(k)})). \end{cases} \tag{2.2}$$

It is worth noting that the second and third steps in (2.1) and (2.2) have been built in such a way that no additional computations of any Jacobian matrices are required. In order to analyse the convergence order of the proposed methods (2.1) and (2.2) we address Theorem 2.1.

Theorem 2.1. *Let $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be sufficiently Fréchet differentiable at each point of an open convex neighbourhood D of $x^* \in \mathbb{R}^n$, that is, a solution of the system $F(x) = 0$. Let us suppose that $F'(x)$ and $F'(y)$ are continuous and nonsingular in x^*. Then, the sequence $\{x^{(k)}\}_{k \geq 0}$ obtained using the iterative methods (2.1) and (2.2) converges to x^* with convergence rates four and six, respectively.*