Applied Mathematics and Computation 266 (2015) 369–373

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A remark on one non-autonomous stochastic Gompertz model with delay

Dianli Zhao*

College of Science, University of Shanghai for Science and Technology, Shanghai 200093, China

ARTICLE INFO

Keywords:
Stochastic Gompertz model with delay
Comparison theorem
Persistence and extinction

ABSTRACT

In this paper, we will point out the errors existing in the proof in [1] given by Jovanovic and Krstic (2014) on applications of the comparison theorem which are key steps to the proof of the main result. Then we give the sufficient conditions for persistence in mean and extinction of the considered model with completely new proofs, where we do not ask for the application of the comparison theorem.

1. Introduction

The Gompertz model has been one of the most popular models in biomathematics. Subjected to environmental randomness, many authors have studied the Gompertz population models with Brown white noises, see e.g. [1–4]. Zou, Li and Wang [5] have investigated the optimal harvesting policy for the Gompertz model driven by Levy noises.

Persistence and extinction are two basic and important topics, which mean that a population system will survive forever or not. So it maybe of interest to all related researchers. When the persistence and extinction are studied for the stochastic population models, the following method is always applied (see [6,7] for example).

- Step I. Establish the estimation results by using the stochastic comparison theorem.
- Step II. Prove the limit: $\limsup_{t\to\infty} \frac{\ln x(t)}{t} \le 0$.
- Step III. Find the sufficient conditions for persistence and extinction of the population.

The above method is always used to prove the persistence and extinction for the stochastic population models without time delay. However, time delay is always necessary when we model the system in reality. An interesting problem occurs naturally whether the established method is still effective. In [1], Jovanovic and Krstic have taken a try by considering a stochastic Gompertz model with time delay.

Jovanovic and Krstic [1] consider a stochastic Gompertz population model with delay of the form

$$dx(t) = r(t)x(t)(\ln K(t) - \ln x(t-\tau))dt + \alpha(t)x(t)dw(t) \tag{1}$$

with the initial data $x_0 = \{\xi(\theta), -\tau \le \theta \le 0\}$ where $\xi(t)$ is continuous function from $[-\tau, 0]$ to R^+, $x(t)$ represents the number of cells/individuals, $r(t) > 0$, $K(t) > 0$ and $\alpha(t) \ge 0$ are all continuous and bounded functions on $[0, \infty)$, $w(t)$ is a one-dimensional standard Brownian motion. Following the above three steps, Jovanovic and Krstic give sufficient conditions for persistence and

* Tel.: +86 21 65691507.
E-mail address: Tc_zhaodianli@139.com

http://dx.doi.org/10.1016/j.amc.2015.05.051
0096-3003/

extinction. Firstly, by use of the comparison theorem for stochastic differential equations, they establish the estimations, see [[1], Theorem 2.2], namely,

$$\phi(t) \le x(t) \le \Phi(t) \text{ for all } t \ge 0 \text{ a.s.} \tag{2}$$

Then, based on the above estimations, they prove the limit result [[1], Theorem 3.1], and persistence and extinction of the population (see [[1], Theorem 3.2 and Theorem 4.1]). One key step for proof of Theorem 2.2 in [1] reads that

$$\text{suppose that } \sup_{t\ge 0}\left| r(t)\ln K(t) - \frac{\alpha^2(t)}{2}\right| \le L \text{ holds for some constant } L > 0, \tag{3}$$

Denote $y(t) = ln(x(t))$ and $r^u = \sup_{t\ge 0}\{r(t)\}$, then

$$y(t) \le A + \int_0^t \left(r^u K(s+\tau)e^{-y(s)} - \frac{\alpha^2(s)}{2}\right) ds + \int_0^t \alpha(s)dw(s) \tag{4}$$

Let $z(t)$ be a solution of the following equation:

$$dz(t) = \left(r^u K(t+\tau)e^{-z(t)} - \frac{\alpha^2(t)}{2}\right) dt + \alpha(t)dw(t) \tag{5}$$

with $z(0) = y(0) = A$, then by using the comparison theorem for stochastic differential equations to compare (4) with (5) to give

$$y(t) \le z(t) \text{ for all } t \ge 0 \text{ a.s.} \tag{6}$$

Similar result based on the comparison theorem can be found in Line 8-13, Page 104 in [1]. Unfortunately, these comparison results do not hold, that is, the conditions of the stochastic comparison theorem are not satisfied (see Section 2 for details).

The main aim of this paper is to point out the errors existing in the proof in [1] given by Jovanovic and Krstic (2014) when they apply the comparison theorem for stochastic differential equations, which are key steps to the proof of the main result. Then we give the modified results with completely new proofs.

2. Analysis on application of the comparison theorem

In [1], the comparison theorem for stochastic differential equation had been used only to compare a stochastic integration inequality (4) with a stochastic differential equations (5) to give the comparison result (6), and the application of the comparison theorem has nothing to do with (1), so we will focus on the expressions (4)–(6). Firstly, we show that the given conditions in [1] do not satisfy the stochastic comparison theorem. The classical stochastic comparison theorem given by Yamada in [8] is as follows.

Theorem 2.1. *[[8], Theorem 1.1 and Theorem 1.3] Consider two real adapted continuous process $x_1(t)$ and $x_2(t)$ which satisfy the following equations, respectively:*

$$dx_i(t) = b_i(t, x_i(t))dt + \beta(t, x_i(t))dw(t), \quad i = 1, 2 \tag{7}$$

with initial value $x_1(0) = x_2(0)$. $b_1(t, x)$, $b_2(t, x)$ and $\beta(t, x)$ are all continuous functions on $[0. +\infty) \times R$. If

$$b_1(t, x) < b_2(t, x) \text{ for all } (t, x) \in [0, +\infty) \times R. \tag{8}$$

and there exists a positive increasing function $\xi(u)$ on $[0, +\infty)$ with $\xi(0) = 0$ such that

$$|\beta(t, x) - \beta(t, y)| \le \xi(|x - y|), \quad x, y \in R \tag{9}$$

and

$$\int_{0^+}^{\infty} \frac{1}{\xi^2(s)} ds = \infty, \tag{10}$$

then $x_1(t) \le x_2(t)$ a.s. for all $t \ge 0$.

Furthermore, if the pathwise uniqueness of solutions holds for both equations, then $x_1(t) \le x_2(t)$ holds by weakening (1.3) to $b_1(t, x) \le b_2(t, x)$ for all $(t, x) \in [0, +\infty) \times R$.

Remark 1. When the stochastic comparison theorem is applied to prove (6), one has to take $y(t) = x_1(t)$ and $z(t) = x_2(t)$. However, it's easy to see that $y(t)$ isn't consistent with the conditions of $x_1(t)$ since $y(t)$ satisfies the inequality (4) while $x_1(t)$ is solution of one of the equations (7). Thus, we conclude that (6) cannot be obtained directly by using the comparison theorem for stochastic differential equations to compare (4) with (5).

In view of the stochastic comparison theorem, there is a lot of nice work, see [9–11] and cited therein. However, all these comparison results are given for the same kind of equations. To the author's knowledge, there are few papers showing applications of the stochastic comparison theorem between different kinds of stochastic expressions. Therefore, to give under what conditions