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Exponential stability of numerical solution to neutral stochastic functional differential equation

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ABSTRACT

Stability of the solution to neutral stochastic functional differential equation (NSFDE) has received a great deal of attention, but there is so far little work on stability of numerical solution. To close the gap, the paper develops new criteria on stability of numerical solutions to linear and nonlinear NSFDEs. We show that the backward Euler–Maruyama(EM) method can reproduce the almost surely exponential stability of the exact solution to highly nonlinear NSFDE, and EM method can preserve the almost surely exponential stability of NSFDE with linear growing coefficients. Two examples are provided to illustrate the main results.

1. Introduction

Neutral stochastic functional differential equations have received a great deal of attention, research efforts have been devoted to stability of the analytical solutions(see [1–6]). Unfortunately, explicit solutions can rarely be obtained, especially highly nonlinear NSFDEs. So numerical solutions have obtained an increasing attention ([7–10]). The existing results mostly are concerned with convergence of numerical solutions to NSFDEs. For example, Wu and Mao [11] studied the convergence of EM approximation for NSFDE with the linear growth coefficients. Milošević [12] studied convergence in probability of the EM approximate solutions to nonlinear neutral SDDEs under the Khasminskii-type conditions. Zhou and Fang [13] established convergence in probability of the EM approximate solutions for highly nonlinear NSFDEs.

Stability theory of numerical solution is one of the key problems in numerical analysis. However, the study on stability of numerical method for neutral stochastic differential systems is relatively scarce due to their technical difficulties, which is the main topic of the present paper. Recently, Milošević [14] showed that almost sure exponential stability of solutions to highly nonlinear neutral stochastic differential equations with time-dependent delay. To the best knowledge of authors, there is no work on the stability of numerical solution to NSFDE. To close the gap, the paper develops new criteria on stability of numerical solution referring to the technique of stability of the analytical solution. In the paper, we first show that nonlinear NSFDE is almost surely exponentially stable, and EM method can preserve almost surely exponential stability of the analytical solution to NSFDE with the linear growth coefficients. We shall also establish a new criterion on almost surely exponential stability of numerical solution to highly nonlinear NSFDE, with the help of the technique used in stability analysis of the exact solution.

The structure of the paper is as follows: In the next section, after introducing some necessary notations and assumptions, we shall prove that the existence-and-uniqueness of the global solution and almost surely exponential stability of the exact solution to NSFDE under polynomial growth conditions. Section 3 proves that EM method can preserve almost surely exponential stability of the analytical solution for NSFDE with the linear growth coefficients. Section 4 shows that the backward EM scheme

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may reproduce almost surely exponential stability to highly nonlinear NSFDEs. In Section 5, one highly nonlinear example is considered to illustrate the main theory, which implies our results are very general, so that a wide class of nonlinear systems obey our criteria.

2. The global solution

Throughout this paper, unless otherwise specified, let $|x|$ be the Euclidean norm in $x \in \mathbb{R}^n$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{trace(A^TA)}$, while its operator norm is denoted by $\|A\| = \sup\{|Ax| : |x| = 1\}$ (without any confusion with $\|\varphi\|$). Let be $\mathbb{R}_+ = [0, \infty)$ and for $\tau > 0$, we shall denote by $C([-\tau, 0]; \mathbb{R}^n)$ the family of continuous functions φ from $[-\tau, 0]$ to $\mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{-\tau\le\theta\le0} |\varphi(\theta)|$. For simplicity, we also have to denote by $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge0}$, satisfying the usual conditions (i.e., it is increasing and right continuous and $\mathcal{F}_0$ contains all P-null sets). Denoted by $C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ the family of all $\mathcal{F}_0$-measurable $C([-\tau, 0]; \mathbb{R}^n)$-valued random variables $\xi = \{\xi(\theta) : -\tau \le \theta \le 0\}$ such that $\sup_{-\tau\le\theta\le0} \mathbb{E}|\xi(\theta)|^2 < +\infty$. Let $w(t)$ be a scalar Brownian motion defined on the probability space.

Consider an n-dimensional neutral stochastic functional differential equation

$$d[x(t) - u(x_t)] = f(x(t), x_t)dt + g(x(t), x_t)dw(t) \tag{2.1}$$

on $t \ge 0$ with initial data $x_0 = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$, $x_t = \{x(t+\theta) : -\tau \le \theta \le 0\}$, which is regarded as a $C([-\tau, 0]; \mathbb{R}^n)$-valued stochastic process. Moreover, $u : C([-\tau, 0]; \mathbb{R}^n) \longrightarrow \mathbb{R}^n$, $f : \mathbb{R}^n \times C([-\tau, 0]; \mathbb{R}^n) \longrightarrow \mathbb{R}^n$,$g : \mathbb{R}^n \times C([-\tau, 0]; \mathbb{R}^n) \longrightarrow \mathbb{R}^n$ are Borel-measurable functions. For the convenience, let $\tilde{x}(t) = x(t) - u(x_t)$.

For the purpose of the stability, we assume that $f(0, 0) = g(0, 0) = 0$ and $u(0) = 0$. Therefore Eq. (2.1) admits a trivial solution $x(t; 0) = 0$ corresponding to the trivial data $x_0 = 0$. In this paper, we shall only discuss almost surely exponential stability of the trivial solution. To guarantee the existence-and-uniqueness and stability of the global solution, we need to impose the following conditions on the drift coefficient f and the diffusion coefficient g.

(H1). (The local Lipschitz condition): For each integer $B \ge 1$, there is a positive constant K_B such that

$$\begin{aligned}&|f(\varphi_1(0), \varphi_1) - f(\varphi_2(0), \varphi_2)|^2 \vee |g(\varphi_1(0), \varphi_1) - g(\varphi_2(0), \varphi_2)|^2\\ &\quad\le K_B(|\varphi_1(0) - \varphi_2(0)|^2 + \|\varphi_1 - \varphi_2\|^2)\end{aligned}$$

for any $\varphi_i(0) \in \mathbb{R}^n$, $\varphi_i \in C([-\tau, 0]; \mathbb{R}^n)$ with $|\varphi_i(0)| \vee \|\varphi_i\| \le B(i = 1, 2)$.

(H2). (The polynomial growth conditions): Assume that there exist positive constants $\alpha, a_0, \hat{a}, \bar{a}, \tilde{a}, b_0, \bar{b}, \tilde{b}$ such that

$$2\langle\tilde{\varphi}(0), f(\varphi(0), \varphi)\rangle \le -a_0|\varphi(0)|^2 + \frac{\hat{a}}{\tau}\int_{-\tau}^0 |\varphi(\theta)|^2 d\theta + \frac{\bar{a}}{\tau}\int_{-\tau}^0 |\varphi(\theta)|^{\alpha+2} d\theta - \tilde{a}|\varphi(0)|^{\alpha+2}, \tag{2.2}$$

$$|g(\varphi(0), \varphi)|^2 \le b_0|\varphi(0)|^2 + \frac{\bar{b}}{\tau}\int_{-\tau}^0 |\varphi(\theta)|^{\alpha+2} d\theta + \tilde{b}|\varphi(0)|^{\alpha+2}, \tag{2.3}$$

for any $\varphi(0) \in \mathbb{R}^n$, $\varphi \in C([-\tau, 0]; \mathbb{R}^n)$, $\tilde{\varphi}(0) = \varphi(0) - u(\varphi)$.

(H3). (The contractive condition): Assume that there exist a constant $\kappa(0 < \kappa < 1)$ such that

$$|u(\varphi) - u(\psi)|^2 \le \frac{\kappa}{\tau}\int_{-\tau}^0 |\varphi(\theta) - \psi(\theta)|^2 d\theta \tag{2.4}$$

for any $\varphi, \psi \in C([-\tau, 0]; \mathbb{R}^n)$, moreover let $u(0) = 0$.

Clearly, $\int_{-\tau}^0 |\varphi(\theta) - \psi(\theta)|^2 d\theta \le \tau \sup_{-\tau\le\theta\le0} |\varphi(\theta) - \psi(\theta)|^2 = \tau\|\varphi - \psi\|$, that is,$|u(\varphi) - u(\psi)|^2 \le \kappa\|\varphi - \psi\|^2$.

Theorem 2.1. *Assume that (H1)–(H3) hold with $a_0 > b_0 + \hat{a}$, $\tilde{a} > \bar{a} + \bar{b} + \tilde{b}$. Then for any initial data $\xi \in C^b_{\mathcal{F}_0}((-\tau, 0]; \mathbb{R}^n)$, there almost surely exists a unique global solution $x(t)$ to Eq. (2.1) on $t > -\tau$.*

Proof. Under (H1), applying the standard truncation technique [see Mao[15],Theorem 3.15, P_{91}] to Eq. (2.1) for any initial data $\xi \in C^b_{\mathcal{F}_0}((-\tau, 0]; \mathbb{R}^n)$, there exists a unique maximal local strong solution to Eq. (2.1) on $-\tau < t < \sigma_e$, where σ_e is the explosion time. To show this solution is global, we only need to show that $\sigma_e = \infty$ *a.s.* Let $h_0 > 0$ be sufficiently large such that $h_0 \ge |\tilde{x}_0|$. For each integer $h \ge h_0$, define the stopping time

$$\sigma_h = \inf\{t \in [0, \sigma_e) : |\tilde{x}(t)| \ge h\}, (h \in \mathbb{N}), \tag{2.5}$$

where throughout this paper, we set $\inf\emptyset = \infty$(as usual, $\emptyset =$ the empty set). Obviously, σ_h is an increasing function with h, so $\sigma_h \to \sigma_\infty \le \sigma_e(h \to \infty)$. If we can show that $\sigma_\infty = \infty$ *a.s.*, then $\sigma_e = \infty$ *a.s.* In other words, we only need to prove that $\mathbb{P}(\sigma_h \le t) \to 0(h \to \infty, t > 0)$.