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Even-order half-linear advanced differential equations: improved criteria in oscillatory and asymptotic properties

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ABSTRACT

We establish some new criteria for oscillation and asymptotic behavior of solutions of even-order half-linear advanced differential equations. We study the case of canonical and the case of noncanonical equations subject to various conditions.

1. Introduction

In the natural sciences, technology, and population dynamics, differential equations find many application fields; see [13]. In recent years, there has been an increasing interest in studying oscillation of various classes of differential equations. We refer the reader to [1–12,14–24] and the references cited therein.

Many authors studied oscillatory behavior of the higher-order differential equation

$$\left((x^{(n-1)}(t))^{\alpha}\right)' + f(x(\tau(t))) = 0.$$

As a special case

$$\left((x^{(n-1)}(t))^{\alpha}\right)' + q(t)x^{\alpha}(\tau(t)) = 0, \tag{1.1}$$

where n is even, α is the ratio of odd positive integers, Agarwal and Grace [2] and Agarwal et al. [5] established the following results.

Theorem 1.1 (See [2, Theorem 3.6]). *Let*

$$q, \tau \in \mathrm{C}([t_0, \infty), \mathbb{R}), \ q(t) \geq 0, \ \text{and} \ \tau(t) \geq t \text{ for } t \geq t_0. \tag{1.2}$$

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If

$$\int_t^\infty q(s)\mathrm{d}s < \infty,$$

$$\liminf_{t\to\infty} \int_t^{\tau(t)} s^{n-2}\left(\int_s^\infty q(u)\mathrm{d}u\right)^{1/\alpha}\mathrm{d}s > \frac{(n-2)!}{\mathrm{e}},$$

and

$$\liminf_{t\to\infty} \int_t^{\tau(t)} [\tau(s)-s]^{n-2}\left(\int_{\tau(s)}^\infty q(u)\mathrm{d}u\right)^{1/\alpha}\mathrm{d}s > \frac{(n-2)!}{\mathrm{e}},$$

then (1.1) *is oscillatory.*

Theorem 1.2 (See [2, Theorem 4.1]). *Let* (1.2) *hold. If for all constants* $\theta \in (0, 1)$,

$$\limsup_{t\to\infty} \frac{t^{n-1}}{(n-1)!}\left[\int_t^\infty q(s)\mathrm{d}s + \frac{\theta\alpha}{2(n-2)!}\int_t^\infty s^{n-2}\left(\int_s^\infty q(v)\mathrm{d}v\right)^{(\alpha+1)/\alpha}\mathrm{d}s\right]^{1/\alpha} > 1,$$

then (1.1) *is oscillatory.*

Theorem 1.3 (See [5, Theorem 2.1, $\sigma(t) = t$]). *Let* (1.2) *hold. If there exists a function* $\rho \in C^1([t_0,\infty),(0,\infty))$ *such that, for all constants* $\theta \in (1, \infty)$,

$$\int^\infty \left[\rho(t)q(t) - \theta\frac{(2(n-2)!)^\alpha}{(\alpha+1)^{\alpha+1}}\frac{(\rho'(t))^{\alpha+1}}{(t^{n-2}\rho(t))^\alpha}\right]\mathrm{d}t = \infty,$$

then (1.1) *is oscillatory.*

Theorem 1.4 (See [5, Theorem 2.3]). *Let* (1.2) *hold,* $\tau \in C^1([t_0,\infty),\mathbb{R})$, *and* $\tau'(t) > 0$ *for* $t \ge t_0$. *If*

$$\limsup_{t\to\infty} t^{\alpha(n-1)}\int_t^\infty q(s)\mathrm{d}s > ((n-1)!)^\alpha,$$

then (1.1) *is oscillatory.*

Grace and Lalli [11] considered oscillation of an even-order equation

$$x^{(n)}(t) + q(t)x(\tau(t)) = 0, \tag{1.3}$$

and obtained the following result.

Theorem 1.5 (See [11, Theorems 2 and 3, $\sigma(t) = t$]). *Let* (1.2) *hold. If there exists a function* $\rho \in C^1([t_0,\infty),(0,\infty))$ *such that*

$$\int^\infty \left[\rho(t)q(t) - \frac{(n-1)!}{2^{3-2n}}\frac{(\rho'(t))^2}{t^{n-2}\rho(t)}\right]\mathrm{d}t = \infty,$$

then (1.3) *is oscillatory.*

Following the papers [2,5,11], we are concerned with an advanced differential equation

$$\left(r(t)(x^{(n-1)}(t))^\alpha\right)' + q(t)x^\alpha(\tau(t)) = 0, \tag{1.4}$$

where $t \ge t_0$, α is the ratio of odd positive integers, $r \in C^1([t_0,\infty),\mathbb{R})$, $r(t) > 0$, $r'(t) \ge 0$, $q, \tau \in C([t_0,\infty),\mathbb{R})$, $q(t) \ge 0$, and $\tau(t) \ge t$. Similar as in the papers by Džurina and Kotorová [9] and Li et al. [16], Eq. (1.4) is called canonical if

$$\int_{t_0}^\infty \frac{\mathrm{d}t}{r^{1/\alpha}(t)} = \infty, \tag{1.5}$$

whereas it is termed noncanonical in the case when

$$\int_{t_0}^\infty \frac{\mathrm{d}t}{r^{1/\alpha}(t)} < \infty. \tag{1.6}$$

By a solution of (1.4) we mean a function $x \in C^{n-1}[T_x,\infty)$, $T_x \ge t_0$, which has the property $r(x^{(n-1)})^\alpha \in C^1[T_x,\infty)$ and satisfies (1.4) on $[T_x,\infty)$. We consider only those solutions x of (1.4) which satisfy $\sup\{|x(t)| : t \ge T\} > 0$ for all $T \ge T_x$ and assume that (1.4) possesses such solutions. A solution of (1.4) is said to be oscillatory if it has arbitrarily large zeros on $[T_x,\infty)$; otherwise, it is termed nonoscillatory. Eq. (1.4) is called oscillatory if all its solutions are oscillatory.

Zhang et al. [23,24] obtained some oscillation criteria for (1.4) in the case $\tau(t) < t$. The natural question now is: *is it possible to establish new oscillation and asymptotic criteria for* (1.4) *in the case where* $\tau(t) \ge t$? Our aim in this paper is to give an affirmative answer to this question. In what follows, all functional inequalities are assumed to hold eventually, that is, they are satisfied for all t large enough.