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A strongly sub-feasible primal-dual quasi interior-point algorithm for nonlinear inequality constrained optimization[☆]

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ABSTRACT

In this paper, a primal-dual quasi interior-point algorithm for inequality constrained optimization problems is presented. At each iteration, the algorithm solves only two or three reduced systems of linear equations with the same coefficient matrix. The algorithm starts from an arbitrarily initial point. Then after finite iterations, the iteration points enter into the interior of the feasible region and the objective function is monotonically decreasing. Furthermore, the proposed algorithm is proved to possess global and superlinear convergence under mild conditions including a weak assumption of positive definiteness. Finally, some encouraging preliminary computational results are reported.

1. Introduction

In this paper, we consider the following nonlinear inequality constrained optimization problem

$$\begin{aligned} &\min\ f(x) \\ &\text{s.t. } g_i(x) \le 0, \quad i \in I = \{1, 2, \ldots, m\}, \end{aligned} \tag{1.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}, i \in I$ are all continuously differentiable. The Karush–Kuhn–Tucker (KKT) first order necessary conditions of optimality for problem (1.1) are

$$\nabla f(x) + \sum_{i\in I} \lambda_i \nabla g_i(x) = 0; \quad \lambda_i g_i(x) = 0,\ \lambda_i \ge 0,\ g_i(x) \le 0,\ i \in I. \tag{1.2}$$

Applying a quasi–Newton iteration to the solution of the equations in (1.2), one can solve the following system of linear equations (SLE) in (d^0, λ^0):

$$Hd^0 + \sum_{i\in I} \lambda_i^0 \nabla g_i(x) = -\nabla f(x); \quad z_i \nabla g_i(x)^T d^0 + \lambda_i^0 g_i(x) = 0,\ i \in I, \tag{1.3}$$

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where H is an estimate of the Hessian of the Lagrangian function $L(x, \lambda) := f(x) + \sum_{i \in I} \lambda_i g_i(x)$, x is the current estimate of a solution x^*, and $x + d^0$ and λ^0 are the next estimate of x^* and the KKT multiplier vector, respectively. It can be shown that if H is positive definite, each z_i is strictly positive, and x satisfies the strict inequality constraints, i.e, $g_i(x) < 0$ for all i, then d^0 is a descent direction for both the object function f and the Lagrangian function $L(\cdot, \lambda^0)$. However, d^0 may not be sufficient feasible for the feasible set of problem (1.1). Thus, to obtain an updated direction d^1 for d^0, by perturbing the right-hand side of the second equation of (1.3) by $-z_i \|d^0\|^\nu$, Panier et al. [1] solve an additionally SLE with the form of

$$Hd + \sum_{i=1}^{m} \lambda_i \nabla g_i(x) = -\nabla f(x); \quad z_i \nabla g_i(x)^T d + \lambda_i g_i(x) = -z_i \|d^0\|^\nu, \quad i \in I, \tag{1.4}$$

where constant $\nu > 2$. Then, the search direction $\bar{d}$ is yielded by a convex combination of d^0 and d^1 so that $\bar{d}$ is both descent and sufficiently feasible. Furthermore, to avoid the Maratos effect [2], a higher order correction direction $\tilde{d}$ is obtained by solving the following linear least square problem (LSP):

$$\min_{d \in \mathbb{R}^n} \{ \|d\|^2/2 : \ g_i(x + \bar{d}) + \nabla g_i(x)^T d = -\psi, i \in I_x \}, \tag{1.5}$$

where I_x is an approximate active set at x, and ψ is a scalar variable.

Nevertheless, the algorithm [1] has two difficulties that need to be further studied. First, the system (1.4) may become ill-conditioned when some multiplier z_i^k corresponding to a nearly active constraint g_i becomes very small. Although the algorithm [1] was proven to converge eventually to KKT points for (1.1), it may get bogged down over a significant number of iterations in the neighborhood of some non-KKT stationary points, i.e., not all multipliers corresponding to those stationary points are nonnegative (these stationary points may be constrained local maxima or constrained saddle points). Second, in the analysis of global convergence, they require an additional assumption that the number of stationary points is finite. Gao et al. [3] attempted to solve the latter difficulty by solving an extra SLE, and assuming that the sequence of the approximate multipliers is bounded.

To further improve the algorithm [1], Bakhtiari and Tits [4] proposed a simple primal-dual feasible interior-point method. Here, a suitable vector of barrier parameters is introduced, i.e., a different barrier parameter for each constraint; to yield a search direction and a correction direction, two SLEs and one LSP need to be solved. Particularly, in the second system, the idea of interior-point methods is adopted to construct the vector of barrier parameters. Without the assumption of isolatedness of the stationary points and the positive definiteness on matrix H_k, the algorithm of [4] achieves global convergence and two-step superlinear convergence. Another improvement of the algorithm [4] is that it allows the initial point to lie on the boundary of the feasible set.

However, at least four problems are worthy of further research on the algorithm [4]. Firstly, the KKT conditions of the LSP can be formulated as a system of linear equalities, but its coefficient matrix is different from those of the previous two SLEs, which makes the computational expense relatively higher. Secondly, the algorithm is only two-step superlinearly convergent. The third problem is that the scale of the SLEs ($n + m$ variables and $n + m$ equations) is large since all the constraints and their gradients are included. At last, the initial point must lie on the feasible set, while computing a feasible point is generally a nontrivial work for some practical problems.

Zhu [5] improved the algorithm [4] and overcame the first problem mentioned above, but the additional assumption of isolatedness of the stationary points and the uniformly positive definiteness on the sequence $\{H_k\}$ are still required.

By further modifying the algorithms [4,5], Jian and Pan [6] proposed a feasible descent primal-dual interior-point algorithm for the solution of problem (1.1). Here, the former three problems stated above are solved successfully, but the fourth one is unsolved, i.e, a feasible point is required to initialize the algorithm. In fact, such a difficulty not only appears here but also in many other methods of feasible type, such as feasible sequential quadratic programming [7] and feasible sequential quadratically constrained quadratic programming [8]. In order to overcome such kind of difficulty in a more general context, Jian and his collaborators proposed a method of strongly sub-feasible directions (MSSFD), see [9, Chapter 2] and [10–13]. The main features of the MSSFD can be described as follows: the initial point can be chosen arbitrarily without using any penalty parameters or penalty functions; the feasibility of a constraint is maintained through the iterations once it is reached, and therefore the number of feasible constraints is nondecreasing; the operations of initialization (Phase I) and optimization (Phase II) can be well unified automatically. Furthermore, if the search directions are constructed elaborately, after finite iterations, the iteration points can all get into the feasible set.

In this paper, based on the primal-dual interior-point algorithm [4] and the idea of the MSSFD, we propose a new algorithm called primal-dual quasi interior-point algorithm. First, motivated by the identification technique of active set [14], we propose a new working set that has less elements. Therefore the associated SLE possesses smaller scale and requires less computational cost than those [4,14].

Second, by combining the new working set with the idea of MSSFD, at each iteration, we solve three or two SLEs with the form of

$$\begin{pmatrix} H_k & A_k \\ Z_k A_k^T & B_k \end{pmatrix} \begin{pmatrix} d \\ \lambda \end{pmatrix} = \begin{pmatrix} -\nabla f(x^k) \\ \mu \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ \mu \end{pmatrix}, \tag{1.6}$$

where the definitions of the matrices H_k, Z_k, B_k and A_k are described in Section 2.

Third, since the analysis of global convergence requires that the components of $Z_k := \mathrm{diag}(z_i^k)$ corresponding to the active constraints are uniformly bounded from below, how to design a suitable updating rule for z^k is of great significance. Actually, one