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A new collocation method for approximate solution of the pantograph functional differential equations with proportional delay

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ABSTRACT

The paper presents a new numerical method for solving functional differential equations with proportional delays of the first and higher orders. The method consists of replacing the initial equation by an approximate equation which has an exact analytic solution with a set of free parameters. These free parameters are determined by the use of the collocation procedure. Some examples are given to demonstrate the validity and applicability of the new method and a comparison is made with the existing results. The numerical results show that the proposed method is of a high accuracy and is efficient for solving a wide class of functional-differential equations with proportional delays including equations of neutral type. The method is applicable to both initial and boundary value problems.

1. Introduction

In this study, we will consider functional differential equations with proportional delays which are usually referred to as pantograph equations

$$u^{(n)} = \sum_{j=1}^{J} \sum_{k=0}^{n-1} P_{j,k}(x) u^{(k)}\left(\alpha_j x\right) + f(x), \quad x \in [0, T] \tag{1}$$

subject to the following initial conditions

$$u^{(k)}(0) = a_k, \quad k = 0, 1 \ldots, n-1. \tag{2}$$

We also consider boundary value problems (BVPs), when the conditions are given at both endpoints of $[0, T]$

$$u^{(i)}(0) = a_i, \quad i = l_1, l_2, \ldots, l_{n_1}, \quad u^{(j)}(T) = b_j, \quad j = k_1, k_2, \ldots, k_{n_2}, \quad n_1 + n_2 = n. \tag{3}$$

The applications of the pantograph equation exist in different fields such as number theory, economy, biology, control, electrodynamics, nonlinear dynamical systems, quantum mechanics, probability theory, astrophysics, cell growth etc [1,8,14,29]. Thus, during the last decades many numerical techniques have been developed in this field.

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A general information on the numerical methods for delay differential equations can be found in [3]. The authors of [16] considered the stability of one-leg θ-methods for the solution of the pantograph equation. The properties of the analytic and numerical solutions of the multi-pantograph equation presented in [26]. In [27] the asymptotical stability of the analytic and numerical solutions with the constant step size for pantograph equations were investigated. The authors of [20] proposed a piecewise approximation with quasi-uniform meshes which corresponds to the collocation method for the pantograph differential equation. In [32] the variational iteration method [10] is applied to solve the generalized pantograph equation. The technique of [32] provides a sequence of functions which converges to the exact solution of the problem and is based on the use of the Lagrange multipliers for identification of optimal value of a parameter in the functional. The aim of [24] is based on the search for a solution in the form of a series with easily computed components. The authors of [21] proposed a numerical approach for delay differential equations based on spectral methods. The Legendre-collocation method is employed there to obtain numerical approximations of the exact solution. The authors of [22] analyzed the convergence properties of the spectral method for pantograph-type differential and integral equations with multiple delays. The Haar wavelets and Haar's product matrix are used in [18] for solving time-varying functional differential equations. A similar technique based on the use of the general Legendre wavelets is presented by Wang [40] to solve time-varying systems. The homotopy perturbation method for solving the generalized (retarded or advanced) pantograph equation under initial value conditions is presented in [46]. In [11,13] the pantograph equation is investigated using the Adomian decomposition method which is based on the search for a solution in the form of a series with easily computed components.

The numerical method based on the Taylor polynomials is introduced in [34,35] for the approximate solution of the linear pantograph equation. The authors of [36] applied the Taylor method to approximate solution of the non-homogenous multi-pantograph equation with variable coefficients. In [37] the authors applied the Taylor method to approximate solution of the non-homogeneous multi-pantograph equation with variable coefficients. The numerical method based on the Taylor polynomials is introduced in [15] for the approximate solution of the pantograph equations with constant and variable coefficients. The collocation method based on the Bernstein polynomials is introduced for the approximate solution of the pantograph-type differential equations in [23] with retarded case or advanced case. Li and Liu [25] gave the sufficient condition that assure the Runge–Kutta methods with a variable step size are asymptotically stable when it is applied to the multi-pantograph equation. Recently, the authors in [9,32,47] have developed the variational iteration method to solve the multi-pantograph delay and the generalized pantograph equations. The Bessel polynomials are used to obtain the approximation solution of a generalized pantograph equation with variable coefficients in [48]. The approximate solutions in the form of series of the Bessel functions of the first kind with unknown coefficients was proposed for solving the system of multi-pantograph equations with mixed conditions in [49]. The discontinuous Galerkin method with the uniform underlying meshes was proposed in [7,19] to delay differential equations with vanishing delay. A numerical method for two-point boundary value problems pantograph equations has been developed in [5]. The method combines Picard's sequence of successive approximations with the trapezoidal quadrature rule and the Birkhoff interpolation procedure. In [6] the Birkhoff interpolation procedure is replaced by the cubic spline interpolation procedure.

A method to approximate the generalized pantograph equation using the Jacobi rational functions was proposed in [12]. The coefficients of the approximation are determined by using the Jacobi rational–Gauss collocation method and the initial conditions (2). The spectral collocation method for neutral functional-differential equations with proportional delays which combines approximation by the shifted Legendre polynomials and collocation procedure (SLC) was developed in [4]. An analogous collocation method based on the Bernoulli operational matrix was developed in [38]. After using Bernoulli operational matrix the pantograph equation is collocated at suitable collocation points. This collocation together with equations comes from initial conditions generate the algebraic system to determinate coefficients of the approximation. The new numerical collocation method, which is called a matrix method, based on the Boubaker polynomials was developed in [2]. The numerical method based on polynomial approximation, using the Hermite polynomial basis was developed in [45]. The Lanczos tau method is applied to find the Chebyshev polynomial approximations for the solutions of pantograph differential equations in [39]. The numerical scheme to solve the pantograph equation developed in [33] consists in expressing of the required approximate solution as a linear combination of the shifted Chebyshev polynomials. With the help of the Chebyshev pantograph operational matrix the problem is reduced to a set of algebraic equations. It is important to note that this technique is demonstrated in application to the pantograph equation with neutral term too. This important type of functional differential equations with proportional delays has the form

$$u^{(n)}(x) = F\left(x, u(rx), u^{(1)}(rx), \ldots, u^{(n)}(rx)\right), \quad 0 < r < 1 \tag{4}$$

The linear version of (4) can be written in the form

$$u^{(n)}(x) = \sum_{k=0}^{n} P_k(x) u^{(k)}(rx) + f(x). \tag{5}$$

The Runge–Kutta methods for a class of neutral infinite delay-differential equations with different proportional delays were studied in [50]. In [41–43] the one-leg θ-methods applied to the nonlinear neutral differential equations with proportional delay like (4) are considered. In [44] this approach was extended to the neutral Volterra delay-integro-differential equations. The numerical method with the use of the Chebyshev cardinal functions for pantograph equation, pantograph equation with neutral term and the multiple-delay Volterra integral equation with a large domain is proposed in [17].