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Nonlocal initial value problems for differential equations with Hilfer fractional derivative☆

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ABSTRACT

In this paper, we discuss the existence of solutions to nonlocal initial value problem for differential equations with Hilfer fractional derivative. To begin with, we establish an equivalent mixed type integral equation for our problem. Then, we utilize basic properties of Hilfer fractional calculus and fixed point methods to derive three fundamental existence results in the weighted space of continuous functions. Some examples are given to illustrate our results.

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1. Introduction

The objective of the paper is to discuss nonlocal initial value problem of the following Hilfer type fractional differential equation:

$$\begin{cases} D_{a^+}^{\alpha,\beta}u(t) = f(t,u(t)), 0 < \alpha < 1, 0 \le \beta \le 1, t \in (a,b], \\ I_{a^+}^{1-\gamma}u(a^+) = \sum_{i=1}^{m} \lambda_i u(\tau_i), \alpha \le \gamma = \alpha+\beta-\alpha\beta < 1, \tau_i \in (a,b], \end{cases} \tag{1}$$

where the two parameter family of fractional derivative $D_{a^+}^{\alpha,\beta}$ denote the left-sided Hilfer fractional derivative of order α and type β, which is a interpolator between Riemann-Liouville and Caputo fractional derivatives. The operator $D_{a^+}^{\alpha,\beta}$ is a generalized of Riemann-Liouville fractional derivative operator, for further details see definitions in Section 2 and Ref. [1]. The operator $I_{a^+}^{1-\gamma}$ denotes the left-sided Riemann-Liouville fractional integral as presented in Section 2. The nonlinear term $f : (a,b] \times \mathbb{R} \to \mathbb{R}$ is a given function, $\tau_i, i = 1, 2, \ldots, m$ are pre-fixed points satisfying $a < \tau_1 \le \cdots \le \tau_m < b$ and $\Gamma(\gamma) \ne \sum_{i=1}^{m} \lambda_i(\tau_i - a)^{\gamma-1}$ where $\Gamma(\gamma) = \int_0^{+\infty} x^{\gamma-1}e^{-x}dx, \gamma > 0$ is the Gamma function and λ_i are real numbers.

Fractional calculus discusses the differentiation and integration of arbitrary order and arises naturally in various areas of applied science and engineering. For more history and basic results on fractional calculus theory, one can see [1–12]. Fractional differential equations have recently been proved to be tools in physical phenomena models and have attracted great attention

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of several researchers. It is a development in the theory and application of fractional differential equations with the Riemann-Liouville fractional derivative or the Caputo fractional derivative, see [13–20] and reference therein. However, there are a few related works on Hilfer fractional derivatives, for the so-called Hilfer fractional derivatives, one can see [21–23].

It seems that Hilfer et al. [1,21] have initially proposed linear differential equations with the new fractional operator: Hilfer fractional derivative and applied operational calculus to solve such simple fractional differential equations. Thereafter, Furati et al. [22,23] extended to study nonlinear problems and presented the existence, nonexistence and stability results for initial value problems of nonlinear fractional differential equations with Hilfer fractional derivative in a suitable weighted space of continuous functions.

In the present paper, we further extend to study existence of solutions to nonlocal initial value problem about nonlinear Hilfer type differential Eq. (1). We adopt some ideas in [22] to establish an equivalent mixed type integral equation:

$$u(t) = \frac{T(t-a)^{\gamma-1}}{\Gamma(\alpha)} \sum_{i=1}^{m} \lambda_i \int_a^{\tau_i} (\tau_i - s)^{\alpha-1} f(s, u(s))ds + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s, u(s))ds, \tag{2}$$

where

$$T := \frac{1}{\Gamma(\gamma) - \sum_{i=1}^{m} \lambda_i (\tau_i - a)^{\gamma-1}}, \tag{3}$$

for Eq. (1) in the weighted space of continuous functions $C_{1-\gamma}[a, b]$ ($\Gamma(\cdot)$ is the Gamma function). However, we emphasize that we choose different fixed methods: Krasnoselskii fixed point theorem, Schauder fixed point theorem and Schaefer fixed point theorem, to derive the existence results for Eq. (1) in $C^{\gamma}_{1-\gamma}[a, b] \subset C^{\alpha,\beta}_{1-\gamma}[a, b]$.

2. Preliminaries and an equivalent mixed type integral equation

At first, we recall some basic definitions of the Riemann-Liouville fractional integral and derivative which will be made up to the Hilfer fractional derivative.

Definition 2.1. (see [5]) The left-sided Riemann-Liouville fractional integral of order $\alpha \in \mathbb{R}^+$ of function $f(x)$ is defined by

$$(I^{\alpha}_{a^+} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}} \, (x > a),$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. (see [5]) The left-sided Riemann-Liouville fractional derivative of order $\alpha \in [n-1, n)$, $n \in \mathbb{Z}^+$ of function $f(x)$ is defined by

$$(D^{\alpha}_{a^+} f)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x \frac{f(t)dt}{(x-t)^{\alpha-n+1}} \, (x > a),$$

where $\Gamma(\cdot)$ is the Gamma function.

Based on differentiating fractional integrals, a generalized definition called Hilfer fractional derivative can be introduced.

Definition 2.3. (see [1]) The left-sided Hilfer fractional derivative of order $0 < \alpha < 1$ and $0 \le \beta \le 1$ of function $f(x)$ is defined by

$$D^{\alpha,\beta}_{a^+} f(x) = \left(I^{\beta(1-\alpha)}_{a^+} D\left(I^{(1-\beta)(1-\alpha)}_{a^+} f\right)\right)(x),$$

where $D := \frac{d}{dx}$.

The Hilfer fractional derivative is considered as an interpolator between the Riemann-Liouville and Caputo derivative, then the following remarks can be presented to show the relation with Caputo and Riemann-Liouville operators.

Remark 2.4. (see [1])

(i) The operator $D^{\alpha,\beta}_{a^+}$ also can be rewritten as

$$D^{\alpha,\beta}_{a^+} = I^{\beta(1-\alpha)}_{a^+} D I^{(1-\beta)(1-\alpha)}_{a^+} = I^{\beta(1-\alpha)}_{a^+} D^{\gamma}_{a^+}, \ \gamma = \alpha + \beta - \alpha\beta.$$

(ii) Let $\beta = 0$, the left-sided Riemann-Liouville fractional derivative can be presented as $D^{\alpha}_{a^+} := D^{\alpha,0}_{a^+}$.
(iii) Let $\beta = 1$, the left-sided Caputo fractional derivative can be presented as ${}^C D^{\alpha}_{a^+} := I^{1-\alpha}_{a^+} D$.

Secondly, we need the following basic work spaces.

Let $0 < a < b < \infty$ and $C[a, b]$ be the Banach space of all continuous functions from $[a, b]$ into $\mathbb{R}$ with the norm $\|x\|_C = \max\{|x(t)| : t \in [a, b]\}$. For $0 \le \gamma < 1$, we denote the space $C_{\gamma}[a, b]$ as

$$C_{\gamma}[a, b] := \{f(x) : (a, b] \to \mathbb{R} \mid (x-a)^{\gamma} f(x) \in C[a, b]\},$$

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