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Ball convergence comparison between three iterative methods in Banach space under hypothese only on the first derivative

Ioannis K. Argyros [a], Santhosh George [b,*]

[a] Department of Mathematical Sciences, Cameron University, Lawton, OK 73505, USA
[b] Department of Mathematical and Computational Sciences, NIT Karnataka, 575 025, India

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ABSTRACT

We present a convergence ball comparison between three iterative methods for approximating a locally unique solution of a nonlinear equation in a Banach space setting. The convergence ball and error estimates are given for these methods under hypotheses only on the first Fréchet derivative in contrast to earlier studies such as Adomian (1994) [1], Babajee et al. (2008) [13], Cordero and Torregrosa (2007) [17], Cordero et al. [18], Darvishi and Barati (2007) [19], using hypotheses reaching up to the fourth Fréchet derivative although only the first derivative appears in these methods. This way we expand the applicability of these methods. Numerical examples are also presented in this study.

1. Introduction

In this study we are interested in comparing the radii of convergence of three iterative methods for approximating a locally unique solution x^* of equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet-differentiable operator defined on a convex subset of a Banach space X with values in a Banach space Y. In various disciplines of mathematics and engineering sciences, there are wide range of problems, that can be formulated in terms of nonlinear operator equation of the form (1.1) [7,10,35,36,40]. The solutions of these equations (1.1) can rarely be found in closed form. Therefore, solutions of these equations (1.1) are approximated by iterative methods. In particular, the practice of numerical functional analysis for finding such solution is essentially connected to Newton-like methods [1–41]. The study about convergence matter of iterative procedures is usually based on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. There exist many studies which deal with the local and semi-local convergence analysis of Newton-like methods such as [1–41].

We compare the radii of convergence of two-step Newton methods defined for each $n = 0, 1, 2, \ldots$ by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= y_n - F'(y_n)^{-1}F(y_n), \end{aligned} \tag{1.2}$$

* Corresponding author. Tel.: +919731161426.
E-mail addresses: iargyros@cameron.edu (I.K. Argyros), nitksanthosh@gmail.com, sgeorge@nitk.ac.in (S. George).

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$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= y_n - F'(x_n)^{-1}F(x_n) \end{aligned} \tag{1.3}$$

and the two-step method

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= x_n - A(x_n)^{-1}F(x_n), \end{aligned} \tag{1.4}$$

where x_0 is an initial point, $A(x_n) = \sum_{i=1}^{3} \alpha_i F'((1-\theta_i)x_n + \theta_i G(x_n))$, $\alpha_i \theta_i \in [0, 1]$, for each $i = 1, 2, 3$ and $\sum_{i=1}^{3} \alpha_i = 1$, $\sum_{i=1}^{3} \alpha_i \theta_i = \frac{1}{2}$, $\sum_{i=1}^{3} \alpha_i \theta_i^2 = \frac{1}{3}$ and $\sum_{i=1}^{3} \alpha_i \theta_i^3 = \frac{1}{6}$. Method (1.2), method (1.3) and method (1.4) are of convergence order four [7,10,35,36,40], three [19] and at least four [13], respectively. In particular, the local convergence of method (1.4) was studied by Babajee et al. [13] when $X = Y = \mathbb{R}^m$ (m is a positive integer). Moreover, method (1.4) was derived using the Adomian decomposition [1] and the 3-node quadrature rule (see also [17,18]). Furthermore, method (1.4) was compared favorably to method (1.2) and method (1.3). The local convergence was shown using hypotheses reaching up to the third Fréchet derivative of F and the Lipschitz continuity of $F^{(3)}$ although only the first Fréchet derivative appears in these methods. The hypotheses on the Fréchet derivatives limit the applicability of these methods. As a motivational example, let us define function F on $X = [-\frac{1}{2}, \frac{5}{2}]$ by

$$F(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

We have that

$$\begin{aligned} F'(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2 \\ F''(x) &= 6x \ln x^2 + 20x^3 - 12x^2 + 10x \end{aligned}$$

and

$$F'''(x) = 6 \ln x^2 + 60x^2 - 24x + 22.$$

Then, obviously, function F does not have bounded third derivative in X. Notice that, in-particular there is a plethora of iterative methods for approximating solutions of nonlinear equations defined on X [1–41]. These results show that if the initial point x_0 is sufficiently close to the solution x^*, then the sequence $\{x_n\}$ converges to x^*. But how close to the solution x^* the initial guess x_0 should be? These local results give no information on the radius of the convergence ball for the corresponding method. We address this question for method (1.2) in Section 2. The same technique can be used for other methods. In the present study, we extend the applicability of these methods by using hypotheses up to the first derivative of function F and contractions. Moreover we avoid Taylor expansions and use instead Lipschitz parameters. This way we do not have to use higher order derivatives to show the convergence of these methods.

The rest of the paper is organized as follows. In Section 2, we present the local convergence of methods (1.2)–(1.4). The numerical examples are presented in the concluding Section 3.

2. Local convergence analysis

We present the local convergence analysis first of method (1.4) then of method (1.2) and method (1.3) in this section. Let $L_0 > 0$, $L > 0$, $M \geq 1$ and $\alpha, \alpha_i, \theta_i \in [0, 1]$, $i = 1, 2, 3$ be given parameters such that $\sum_{i=1}^{3} \alpha_i = 1$ and $\sum_{i=1}^{3} \alpha_i (1 - \theta_i + \alpha\theta_i) \neq 0$. It is convenient for the local convergence analysis of the method (1.4) that follows to introduce some scalar functions and parameters. Define function g_1 on the interval $[0, \frac{1}{L_0})$ by

$$g_1(t) = \frac{Lt}{2(1 - L_0 t)} \tag{2.1}$$

and parameters r_1, a, r_p by

$$r_1 = \frac{2}{2L_0 + L}, \quad a = L_0 \sum_{i=1}^{3} \alpha_i (1 - \theta_i + \alpha\theta_i), \quad r_p = \frac{1}{a}. \tag{2.2}$$

Moreover, define function p on $[0, +\infty)$ by

$$p(t) = at. \tag{2.3}$$

Then, we have that

$$0 < r_1 < \frac{1}{L_0} \leq r_p, \tag{2.4}$$

$g_1(r_1) = 1$, $p(r_p) = 1$, and $0 \leq g_1(t) < 1$ for each $t \in [0, r_1)$. Furthermore, define functions g_2 and h_2 on the interval $[0, r_1)$ by

$$g_2(t) = \frac{1}{2(1 - L_0 t)} \left(L + \frac{2ML_0(1 + a)}{1 - p(t)} \right) t \tag{2.5}$$