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An efficient three-step method to solve system of nonlinear equations

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ABSTRACT

In this paper, we suggest a sixth order convergence three-step method to solve system of nonlinear equations. Every iteration of the method requires two function evaluations, two first Fréchet derivative evaluations and two matrix inversions. Hence, the efficiency index is $6^{1/(2n+6n^2+\frac{4}{3}n^3)}$, which is better than that of other sixth order methods. The advantages of the method lie in the feature that this technique not only achieves an approximate solution with high accuracy, but also improves the calculation speed. Also, under several mild conditions the convergence analysis of the proposed method is provided. An efficient error estimation is presented for the approximate solution. Numerical examples are included to demonstrate the validity and applicability of the method and the comparisons are made with the existing results.

1. Introduction

System of nonlinear equations is ubiquitous in many areas of applied mathematics and plays vital role in a number of applications such as science and engineering. Most physical problems, such as biological applications in population dynamics and genetics where impulses arise naturally or caused by control, can be modeled by nonlinear equations or a system of those. The system of nonlinear equations is usually difficult to solve analytically; so a numerical method is needed. Construction of iterative methods to approximate solution of system of nonlinear equations is one of the most important tasks in the applied mathematics.

Consider the system of nonlinear equations

$$F(x) = 0, \qquad F : D \to \mathbb{R}^n, \tag{1}$$

in which $F(x) = (f_1(x), f_2(x), \ldots, f_n(x))$ is a Fréchet differentiable function, $D \subseteq \mathbb{R}^n$, and $x = (x_1, x_2, \ldots, x_n)$ is an unknown vector. Suppose that $F(x) = 0$ has a solution $a \in D$.

Newton method is undoubtedly the most famous iterative method to find a by using

$$x^{(k+1)} = x^{(k)} - \frac{F(x^{(k)})}{F'(x^{(k)})}, \tag{2}$$

that converges quadratically in some neighborhood of a [1–3]. Hereafter, to simplicity of notation, we use $1/F'(x^{(k)})$ instead of $(F'(x^{(k)}))^{-1}$. In recent years, several iterative methods have been proposed to improve the order of convergence and efficiency

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of Newton method (2) to solve the system of nonlinear equations (1), by using essentially Taylor's polynomial, decomposition, homotopy perturbation method, quadrature formulas and other techniques [3–10,12–16].

In this paper, we introduce an iterative three-step method to solve (1). It is proved that our method is sixth order convergence and every iteration of it requires two function evaluations, two first Fréchet derivative evaluations and two matrix inversions. Also, we show that the efficiency index is $6^{1/(2n+6n^2+\frac{4}{3}n^3)}$, which is better than that of other sixth order methods. Several numerical examples are given to illustrate the efficiency and the performance of the new iterative method. The obtained results suggest that this new improvement technique introduces a promising and powerful tool for solving system of nonlinear equations.

This paper is organized as follows. In Section 2, we provide our new method to solve (1). It is proved that the method is sixth order convergence and efficiency analysis will be discussed. In Section 3, the proposed method is applied to several types of examples, and comparisons are made with the existing numerical solvers that were reported in other published works in the literature. Finally, some conclusions are given in Section 4.

2. Three-step iterative algorithmdescription

Before discussing the numerical solution of a system given in the form (1), we need to consider some facts.

Definition 2.1. Let $\{x^{(k)}\}_{k\geq 0}$ be a sequence in $\mathbb{R}^n$, which converge to a. Then, the sequence $\{x^{(k)}\}_{k\geq 0}$ is said to convergence of order p to a if there exist a constant c and a natural number N such that $\|x^{(k+1)}-a\| \leq c\|x^{(k)}-a\|^p$, for all $k > N$.

Definition 2.2. Let a be a zero of function $F(x)$ and suppose that $x^{(k-1)}$, $x^{(k)}$ and $x^{(k+1)}$ are three consecutive iterations close to a. Then, the computational order of convergence ρ (denoted by COC) is defined by

$$\rho = \frac{\ln(\|x^{(k+1)}-a\|/\|x^{(k)}-a\|)}{\ln(\|x^{(k)}-a\|/\|x^{(k-1)}-a\|)}.$$

It is well-known that the computational order of convergence ρ can be approximated by means of

$$\rho \approx \frac{\ln(\|x^{(k+1)}-x^{(k)}\|/\|x^{(k)}-x^{(k-1)}\|)}{\ln(\|x^{(k)}-x^{(k-1)}\|/\|x^{(k-1)}-x^{(k-2)}\|)},$$

which was used by Cordero and Torregrosa [14].

Definition 2.3. Let $e^{(k)} = x^{(k)} - a$ be the error in the kth iteration of an iterative method. We call the relation

$$e^{(k+1)} = Ce^{(k)^p} + O(e^{(k)^{p+1}})$$

as the error equation.

If we obtain the error equation for any iterative method, then the value of p is its order of convergence.

For the sake of improving the local order of convergence, many modified methods have been proposed in open literature. In the sequel, we mention some three-step methods.

Cordero and Torregrosa [14] provided the following fifth order convergence method, in which every iteration requires two function evaluations, two first Fréchet derivative evaluations and three matrix inversions:

$$\begin{aligned} y^{(k)} &= x^{(k)} - \frac{F(x^{(k)})}{F'(x^{(k)})}, \\ z^{(k)} &= x^{(k)} - \frac{2F(x^{(k)})}{[F'(y^{(k)}) + F'(x^{(k)})]}, \\ x^{(k+1)} &= z^{(k)} - \frac{F(z^{(k)})}{F'(y^{(k)})}. \end{aligned} \tag{3}$$

Sharma and Gupta [13] provided the following method with fifth order convergence, in which every iteration requires two function evaluation, two first Fréchet derivative evaluations and two matrix inversions:

$$\begin{aligned} y^{(k)} &= x^{(k)} - \frac{1}{2}\frac{F(x^{(k)})}{F'(x^{(k)})}, \\ z^{(k)} &= x^{(k)} - \frac{F(x^{(k)})}{F'(y^{(k)})}, \\ x^{(k+1)} &= z^{(k)} - \left[\frac{2}{F'(y^{(k)})} - \frac{1}{F'(x^{(k)})}\right]F(z^{(k)}). \end{aligned} \tag{4}$$

Cordero et al. [15] provided the following method with sixth order convergence, in which every iteration requires two function evaluations, two first Fréchet derivative evaluations and two matrix inversions: