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Wavelet based quasilinearization method for semi-linear parabolic initial boundary value problems

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ABSTRACT

In this paper, numerical methods based on quasilinearization and Haar and Legendre wavelets to solve a class of semi linear parabolic initial boundary value problem (SPIBVP) have been presented. The Haar and Legendre wavelet methods have been successfully combined with quasilinearization to solve SPIBVP efficiently. The presented numerical scheme has been illustrated using appropriate examples including Fisher equation and the obtained results show that the proposed numerical scheme is robust and easy to apply.

1. Introduction

This paper discusses the numerical method based on wavelet for semi linear parabolic initial boundary value problem(SPIBVP)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + h(x,t,u) \quad \text{in } Q, \quad u|_{\partial_p Q} = \phi, \tag{1.1}$$

where $Q = (0,1) \times (0,T)$ and $\partial_p Q = \partial Q \backslash ((0,1) \times \{T\})$ denotes the parabolic boundary of Q. Here $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and ϕ is the restriction of u on $\partial_p Q$ where $\Phi \in C^{2,1}(\overline{Q})$. Eq. (1.1) represents various mathematical models in mathematical biology, plasma physics and quantum mechanics, to name a few. Considerable attention has been directed towards the development numerical scheme for partial differential equation using operational matrix wavelet methods [1,3–5,12–14,21,22,24]. This method has been systematically studied for linear partial differential equations [1,12,13,21,22], however very few works have been done to solve nonlinear partial differential equations [3–5]. It is also observed that the operational matrix wavelet methods for the non linear partial differential equations in the recent literature fall into two groups: methods for initial value problems [14,24] and the methods for initial and boundary value problems [3–6,15,17]. By assuming the existence and uniqueness of solution as well as the convergence of the quasilinearization scheme, classical quasilinearization based operational matrix wavelet method for various types of ordinary and partial differential equations are studied in [7–9,17–19] and [3,5,6,10,15], respectively. For the time dependent nonlinear partial differential equations, with initial and boundary condition, most of the wavelet based techniques are used only for approximating derivatives with respect to space variables. The time derivatives are always approximated using finite difference approach. In the present work, a new numerical scheme to solve a class of SPIBVPs has been proposed with systematic convergence analysis for quasilinearization. However, in contrast to the methods discussed in [3,5,6,10,15], the new scheme approximate even the derivatives with respect to time using wavelet techniques. Two numerical schemes have been developed by combining classical quasilinearization with two types of wavelets, namely Haar and Legendre

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wavelets. Numerical simulations show that the proposed approach obtain better accuracy than the results in recent literature. The convergence analysis for quasilinearization generalises a recent result of Lakshmikantham et al. [11] as well as simplifies the result of Buică and Precup [2].

The organization of the paper is as follows. In Section 2, we provide a generalised version of the recent result of Lakshmikantham et al. [11]. This section also provides the existence and uniqueness of the solution of SPIBVP and the convergence of the generalised quasilinearization method. Section 3 explains the extension of Haar and Legendre wavelet collocation methods in combination with quasilinearization for SPIBVP. The proposed methods have been illustrated in Section 4 by applying to various examples including Fisher and Newell–Whitehead–Segel type equations. The obtained numerical results are also compared with other numerical results obtained in [15,20,25] using finite difference based Haar wavelet method(FHWM), variational iterative method(VIM), uniform cubic B-spline (UCBS), extended cubic uniform B-spline (ECBS), Trigonometric cubic B-spline (TCBS) and differential quadrature method. We conclude the discussion in Section 5, by stating the merits of the proposed method.

2. Quasilinearization

In this section, we generalise the proof of an existence and uniqueness theorem as well as convergence analysis of [11], for the SPIBVP. Throughout this paper we assume that $E = (E, \leq, \|\cdot\|)$ is an ordered Banach space with order cone E_+. In [11], Lakshmikantham et al. studied an interesting version of fixed point theorem for the operator equation $Tx = x$ where $T: E \to E$, via quasilinearization and its application to SPIBVP. The result presented in [11] is based on the assumption that the operator $u \to T'_u v$ is increasing in u for all $v \in E_+$. However, in the present work, quadratic convergence of the iterative procedure has been proved by relaxing the monotonicity condition assumed in [11]. Throughout this paper, T is decomposed as sum of the continuous operators F and G defined on E. The generalised version of Lakshmikantham et al. [11] result can be stated as follows:

Theorem 2.1. *Let E be an ordered Banach space with a normal order cone E_+. Assume that $T: E \to$* E satisfies the following hypotheses:

1. *$F, G: [v_0, w_0] \to E$ are compact; $\exists\, v_0, w_0 \in E$ such that $v_0 \leq Tv_0, Tw_0 \leq w_0$ and $v_0 \leq w_0$;*
2. *The Frechet derivative F'_u and G'_u exist for every $u \in [v_0, w_0]$; $u \to F'_u v$ and $u \to G'_u v$ are increasing and decreasing, respectively, on $[v_0, w_0]$ for all $v \in E_+$;*
3. $$Fu_0 - Fu_1 \leq F'_{u_0}(u_0 - u_1) \quad \text{whenever} \quad v_0 \leq u_0 \leq u_1 \leq w_0. \tag{2.1}$$
   $$Gu_0 - Gu_1 \leq G'_{u_1}(u_0 - u_1) \quad \text{whenever} \quad v_0 \leq u_0 \leq u_1 \leq w_0. \tag{2.2}$$
4. *$(I - F'_v - G'_w)^{-1}$ exists and it is a bounded positive operator for all $v, w \in [v_0, w_0]$. Then for $n \in \mathbb{N}$, relations*

$$\begin{aligned} v_{n+1} &= Tv_n + (F'_{v_n} + G'_{w_n})(v_{n+1} - v_n) \\ w_{n+1} &= Tw_n + (F'_{v_n} + G'_{w_n})(w_{n+1} - w_n) \end{aligned}$$

define an increasing sequence (v_n) and a decreasing sequence (w_n) which converges to the solutions of the operator equation $Tx = x$. These fixed points are equal if $Tu_1 - Tu_0 < u_1 - u_0$ for all $v_0 \leq u_0 < u_1 \leq w_0$.

Proof. We first prove that v_n, w_n exists for all $n \in \mathbb{N}$ and satisfy

$$v_0 \leq v_1 \leq \cdots \leq v_n \leq w_n \leq w_{n-1} \leq \cdots \leq w_1 \leq w_0 \tag{2.3}$$

We prove this by induction. For $n = 1$, from the definition of v_1, we have $v_1 = (I - F'_{v_0} - G'_{w_0})^{-1}(T - F'_{v_0} - G'_{w_0})v_0$. Hence v_1 exists. Similarly it is easy to verify that w_1 exists. We will show $v_1 \geq v_0$. Let $p = v_1 - v_0$.

$$\begin{aligned} p &= Tv_0 + F'_{v_0}(v_1 - v_0) + G'_{w_0}(v_1 - v_0) - v_0 \\ (I - F'_{v_0} - G'_{w_0})p &\geq 0. \end{aligned}$$

Thus $p \geq 0$. We get $v_1 \geq v_0$. Similarly it can be shown that $w_1 \leq w_0$. Let $p = v_1 - w_1$. Then,

$$\begin{aligned} p &= Tv_0 - Tw_0 + F'_{v_0}(v_1 - v_0 - w_1 + w_0) + G'_{w_0}(v_1 - v_0 - w_1 + w_0) \\ (I - F'v_0 - G'_{w_0})p &= Fv_0 - Fw_0 + Gv_0 - Gw_0 + F'_{v_0}(w_0 - v_0) + G'_{w_0}(w_0 - v_0) \\ (I - F'v_0 - G'_{w_0})p &\leq F'_{v_0}(v_0 - w_0) + G'_{w_0}(v_0 - w_0) + F'_{v_0}(w_0 - v_0) + G'_{w_0}(w_0 - v_0) \end{aligned}$$

Thus $p \leq 0$. Hence $v_1 \leq w_1$. Suppose now that v_j, w_j exist for some $j > 0$ and that

$$v_0 \leq v_1 \leq \cdots \leq v_j \leq w_j \leq w_{j-1} \leq \cdots \leq w_1 \leq w_0 \tag{2.4}$$

We have $v_{j+1} = Tv_j + (F'_{v_j} + G'_{w_j})(v_{j+1} - v_j)$. Thus $v_{j+1} = (I - F'_{v_j} - G'_{w_j})^{-1}(T - F'_{v_j} - G'_{w_j})v_j$. Consequently v_{j+1} exists. Similarly we can show that w_{j+1} exists. Let $p = v_j - v_{j+1}$.