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A Fourier error analysis for radial basis functions and the Discrete Singular Convolution on an infinite uniform grid, Part 1: Error theorem and diffusion in Fourier space

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ABSTRACT

On an infinite grid with uniform spacing h, the cardinal basis $C_j(x;\ h)$ for many spectral methods consists of translates of a "master cardinal function", $C_j(x;\ h) = C(x/h - j)$. The cardinal basis satisfies the usual Lagrange cardinal condition, $C_j(mh) = \delta_{jm}$ where δ_{jm} is the Kronecker delta function. All such "shift-invariant subspace" master cardinal functions are of "localized-sinc" form in the sense that $C(X) = \mathrm{sinc}(X)s(X)$ for a localizer function s which is smooth and analytic on the entire real axis and the Whittaker cardinal function is $\mathrm{sinc}(X) \equiv \sin(\pi X)/(\pi X)$. The localized-sinc approximation to a general $f(x)$ is $f^{\mathrm{localized-sinc}}(x; h) \equiv \sum_{j=-\infty}^{\infty} f(jh)\, s([x - jh]/h)\, \mathrm{sinc}([x - jh]/h)$. In contrast to most radial basis function applications, matrix factorization is unnecessary. We prove a general theorem for the Fourier transform of the interpolation error for localized-sinc bases. For exponentially-convergent radial basis functions (RBFs) (Gaussians, inverse multiquadrics, etc.) and the basis functions of the Discrete Singular Convolution (DSC), the localizer function is known exactly or approximately. This allows us to perform additional error analysis for these bases. We show that the error is similar to that for sinc bases except that the localizer acts like a diffusion in Fourier space, smoothing the sinc error.

1. Introduction

The sinc pseudospectral method is an exponentially-convergent and easily programmed tool to approximate smooth functions and solve differential equations on an unbounded interval ($x \in \mathbb{R}$) with uniform grid spacing h [23,35,36]. The interpolation points (for this basis and all bases throughout this article) are

$$x_j = jh, \qquad j = 0, \pm 1, \pm 2, \ldots \infty \tag{1}$$

and the approximation to a function $f(x)$ is

$$f(x) \approx f^{\mathrm{sinc}}(x) \equiv \sum_{j=-\infty}^{\infty} f(jh)\, \mathrm{sinc}([x - jh]/h) \tag{2}$$

where $\mathrm{sinc}(X) \equiv \sin(\pi X)/(\pi X)$.

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The sinc basis is a "cardinal" or "Lagrange" basis in the sense that, defining

$$C_j(x;h) \equiv \mathrm{sinc}(x/h - j) \tag{3}$$

$$C_j(x_i) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \quad \text{all integer } i, j \tag{4}$$

The cardinal property implies that the coefficients of the cardinal series are simply the "grid point values" or "samples" of $f(x)$, the set $f(x_j)$.

The sinc basis for a given grid spacing h spans a "shift-invariant subspace" in the sense that all the cardinal functions are *translations* of a single "master cardinal function" $C(X)$, which is notationally distinguished by the lack of a subscript

$$C_j(x;h) \equiv C([x - jh]/h) \tag{5}$$

where $C(X) \equiv \mathrm{sinc}(X)$ for the sinc basis.

Note that the cardinal function for $h \neq 1$ is just the *dilation* of the cardinal function for unit h. When the grid points are uniformly spaced, it is always possible to rescale the spatial coordinate to unit spacing without changing the accuracy of the approximation or the cardinal series coefficients

$$X = x/h \qquad \Leftrightarrow \qquad x = h \tag{6}$$

We shall simplify some formulas by giving them for $h = 1$, indicated by using X and K as the spatial coordinate and Fourier transform argument, respectively. There is no loss of generality because of this change of coordinate.

Although spectrally accurate, the sinc method has the large disadvantage that its differentiation matrices are *dense* and it is not possible to circumvent dense matrix manipulations by using the Fast Fourier Transform, which is inapplicable to the sinc basis. However, a wide variety of other "shift-invariant" bases are available.

The sinc basis is unusual in that it is always written in sinc form. Other spectral bases are usually defined by sets of functions that lack the cardinal property. However, it is always possible with *any* basis set and interpolation point set to take linear combinations C_j of the basis functions so that the new basis functions do satisfy (5).

In order to satisfy the cardinal property (for unit grid), the master cardinal function must vanish at all integers except the origin. This implies that all shift-invariant uniform cardinal bases can be written in sinc-factored form as

$$C(X) \equiv \mathrm{sinc}(X)\, s(X) \tag{7}$$

where $s(X)$ is a "localizer" function that is analytic for all real x with $s(0) = 1$.

Below, we shall prove a theorem that explicitly gives the error of the interpolating approximation for *general* $s(X)$. Later, we shall *specialize* to a couple of particular classes of shift-invariant, uniform grid bases (DSC and RBF, defined below) to analyze the approximation error more precisely.

Wei and his collaborators have published an extensive series of articles about a modified sinc pseudospectral method that they dubbed the Discrete Singular Convolution (DSC). This is identical to the standard sinc expansion except that the basis functions are *localized* by the substitution

$$\mathrm{sinc}(X) \rightarrow\; s(X)\,\mathrm{sinc}(X) \tag{8}$$

where $s(X)$ is a user-chosen localizer function that decays rapidly as $|X| \to \infty$. Because of the $s(X)$ factor, the modified master cardinal function $C(X) \equiv s(X)\,\mathrm{sinc}(X)$ now decays rapidly away from its peak at the origin. This makes it possible to *truncate* the DSC differentiation matrices to *sparse* matrices. Wei usually chooses the localizer to be a Gaussian, $s(X) \equiv \exp(-X^2/L^2)$ for some positive constant L [1,4,14,15,17,37], but our results are general, and only restricted to Gaussian-DSC where explicitly noted. The Gaussian-localized basis was independently invented as the "sinc-Gaussian" interpolation [34]. Whatever the name, some DSC/sinc-Gaussian convergence and error theory can be found in [25,28–31,34].

Radial basis functions (RBFs) are a popular method for multidimensional interpolation on irregular or scattered grids [10,11,33,38] and for solving differential equations [13,16,19–22,26,27,39]. RBFs seem at first glance to have little connection with the DSC scheme. In any number of dimensions d, RBF basis functions are of the form :

$$\phi_j \equiv \phi\left(||\vec{x} - \vec{x}_j||_2\right) \qquad \vec{x} \in R^d \tag{9}$$

for some univariate function $\phi(r)$ and some set of N points $\vec{x}_j$, which are called the "centers". (Many species of $\phi(r)$ have been used in the literature as reviewed in [18]). The error falls exponentially with N for smooth $f(\vec{x})$ and certain choices of $\phi(r)$; these "spectrally accurate" RBFs contain a "shape parameter" or "relative inverse width" α. We shall write the various RBF species ϕ as $\phi([\alpha/h]x)$. The user-choosable constant α is the "shape parameter " or "relative inverse width" (these terms are synonyms). The RBF width is written as the ratio α/h because only the width *relative to the grid spacing* is significant.

The RBF basis functions can be recombined into cardinal bases, but unfortunately, the localizers equivalent to the standard RBF species $\phi(r)$ are not known in *exact* explicit form. However, as shown in [7,24], the cardinal function localizer for Gaussian RBFs is, to very accurate *approximation*,

$$s(X) = \frac{\alpha^2 X}{\sinh(\alpha^2 X)}\left\{1 + O(\exp(-2\pi^2/\alpha^2))\right\} \quad \text{(Gaussian RBF localizer)} \tag{10}$$