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Analytical solutions for water pollution problems using quasi-conformal mappings

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ABSTRACT

In this work a straightforward procedure to find exact solutions and point symmetries admitted by nonlinear partial differential equations is presented. This method avoids the explicit computation of the infinitesimals via determining equations, which in some cases are often more difficult to solve than the target equation itself. A preliminary result is obtained by solving the Burgers equation without using the Cole–Hopf transformation.

1. Introduction

The methods based on Lie group analysis, conceived about the 1870s and widely exploited by the Russian school along more than 6 decades, mainly by L. Ovsyannikov, N. Ibragimov and coworkers [5], have been still more successfully employed in the last 20 years to solve nonlinear partial differential equations using software for symbolical calculus.

The symbolical computation packages were initially employed to handle large amount of analytical operations required to deduce and solve the so-called determining equations, an auxiliary system of linear partial differential equations whose solution furnishes the coefficients of the Lie group generators [3]. It occurs that this task, which is the most expensive step in Lie group analysis, recently reveals to be not obligatory for some practical purposes. After the developments introduced by Nikitin [1,2,6,7], and other works which establish relevant connections between symmetries, commutation relations, Bäcklund transformations and differential constraints [4,9], the original Lie method has suffered some interesting simplifications and improvements from the operational point of view. The computational codes produced from these new formulations are small, and the time processing required to obtain exact solutions to partial differential equations have been drastically reduced.

In certain cases for which the determining equations required to obtain the coefficients of the corresponding generators are much more difficult to solve than the own target equation, such as the unsteady two dimensional Helmholtz equation, the application of the original Lie method becomes prohibitive. Fortunately, in practice, many nonlinear problems in transport phenomena can be easily solved by means of coarse mesh formulations. Hence, it is often more convenient to prescribe for each node an exact solution containing a small number of arbitrary parameters than to search for a more general solution which eventually dispenses the discretization of the domain. In these cases, it is not necessary to find all the generators of the corresponding symmetry group, but only some symmetry expressed in explicit form, in order to obtain an exact solution whose arbitrary elements fulfill the restrictions locally prescribed.

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In this work a straightforward method to obtain exact solutions for Burgers-type equations is presented. The solutions obtained using this scheme can be employed in relatively large subdomains, providing a significant reduction in the time processing required to solve the system of algebraic equations arising from the prescriptions of boundary conditions and ensure the continuity at the interfaces of the nodes. The method is based on a principle which is essentially analogous to the underlying idea of classical Lie symmetries. When an exact solution of a given partial differential equation is perturbed by a suitable function with small amplitude, a new exact solution can be obtained. In some sense, this basic idea is closely related to the concept of point symmetries admitted by a differential equation [3,5,8], although neither Lie groups nor direct substitution schemes must be explicitly used to find such symmetries. In this preliminary study, the proposed formulation is applied to obtain exact solutions to the Burgers equation via split induced by scale, a procedure which also resembles some perturbation schemes. The main purpose of this previous analysis is to establish some fundamental features and limitations of a wider class of methods which possibly arises from this apparently naive idea. The ultimate goal of the proposed formulation is to obtain, in future works, exact solutions for the unsteady two-dimensional Navier–Stokes and Helmholtz equations. Once obtained exact solutions to these equations, the mechanisms responsible for the onset of turbulence will be readily identified in their structures. From these solutions, the major result yet expected is a formal proof that the fluctuations arises naturally as a consequence of the presence of nonlinear and high order terms, which acts simultaneously as a turbulence generator and a high wavenumber signal amplifier. This effect is responsible by the production of the whole turbulence spectrum, as well as by the Kolmogorov cascade, a feature which can be easily verified when dealing with formal solutions of the Helmholtz and Navier–Stokes-type equations.

There are some particular limitations of the proposed method which must be stressed in advance. The method is not applicable to ordinary differential equations and cannot be employed to solve partial linear and quasilinear differential equations. However, it must be emphasized that the methods were proposed to circumvent the specific difficulties which arises in solving nonlinear partial differential equations for which fluctuations being not only regarded as a noise, but plays a central role in the evolution of the physical scenarios.

At this point, another important question must be clarified. The authors do not ignore the fact that nowadays the Burgers equation is not difficult to solve via analytical methods. Despite the fact that any exact solution of the heat equation can be easily mapped into a solution of the Burgers equation by applying a nonlinear operator (the Cole–Hopf transformation), this equation is a very convenient starting point to illustrate the main features of the proposed method, as well as to provide a crucial information about the onset of turbulence which yet arises in one dimensional models.

2. Description of the method

In order to elucidate the former arguments with a preliminary example, a simple exact solution to the Burgers equation is obtained. This solution is used to produce a new exact solution containing more arbitrary elements. These arbitrary elements allows the new solution to satisfy a wider set of boundary conditions, or even other restrictions eventually imposed in order to ensure the continuity of the coarse mesh solution along the desired domain.

2.1. Obtaining a closed-form solution

Suppose that no exact solutions to the equation

$$\frac{\partial f}{\partial t} + f.\frac{\partial f}{\partial x} - \nu\frac{\partial^2 f}{\partial x^2} = 0 \tag{1}$$

are known. If one prescribes a solution in the form

$$f = a + \varepsilon\, b \tag{2}$$

where a and b are unknown functions of the independent variables, and ε stands for a small parameter, Eq. (1) becomes

$$\frac{\partial a}{\partial t} + \varepsilon\frac{\partial b}{\partial t} + (a+\varepsilon b)\left(\frac{\partial a}{\partial x} + \varepsilon\frac{\partial b}{\partial x}\right) - \nu\left(\frac{\partial^2 a}{\partial x^2} + \varepsilon\frac{\partial^2 b}{\partial x^2}\right) = 0 \tag{3}$$

The perturbation parameter induces a split by scale, where the coefficients of the powers 0, 1 and 2 in ε, namely

$$\frac{\partial a}{\partial t} + a\frac{\partial a}{\partial x} - \nu\frac{\partial^2 a}{\partial x^2} \tag{4}$$

$$\frac{\partial b}{\partial t} + a\frac{\partial b}{\partial x} + b\frac{\partial a}{\partial x} - \nu\frac{\partial^2 b}{\partial x^2} \tag{5}$$

$$b\left(\frac{\partial}{\partial x}b\right) \tag{6}$$

are null. This system of auxiliary equations produces a particular exact solution to (1). From (6) we get $b = \beta(t)$, which substituted in (5) furnishes

$$\frac{d\beta}{dt} + \beta\frac{\partial a}{\partial x} = 0 \tag{7}$$