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Injective coloring of some graph operations ☆

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ABSTRACT

An injective coloring of a graph G is a vertex coloring such that any two vertices with a common vertex receive distinct colors. The injective chromatic number $\chi_i(G)$ of a graph G is the least k such that there is an injective k-coloring. Graph operations are important methods for constructing new graphs, and they play key roles in the design and analysis of networks. In this study, we give some sharp bounds (or exact values) of graph operations, including the Cartesian product, direct product, lexicographic product, union, join, and disjunction of graphs.

1. Introduction

In this study, we consider simple, finite and undirected graphs. We employ the terminology and notations used by Bondy and Murty [1]. Excluding classical vertex coloring and edge coloring, many types of coloring are studied, such as list coloring, acyclic coloring, and star coloring. In addition, rainbow connection and rainbow vertex-connection are new types of coloring, as described in a recent survey [12] and other studies [2,6,7,9–11]. In the present study, we consider injective coloring.

Let G be a graph. The size of G is the number of vertices of G, which is denoted by $|V(G)|$. A vertex-coloring of a graph is a mapping from its vertex set onto the set of natural numbers. The adjacent vertices may be colored with the same color. An *injective coloring* of a graph G is a vertex-coloring such that any two vertices with a common vertex receive distinct colors. The minimum number of colors needed by G to be injectively colored is the *injective chromatic number* of G, which is denoted by $\chi_i(G)$. Note that this type of coloring is not necessarily proper. Clearly, $\Delta(G) \leq \chi_i(G) \leq |V(G)|$ (as usual, $\Delta(G)$ is the maximum degree of a vertex of G).

The injective coloring of a graph was introduced by Hahn et al. in [4], in which they proved the inequality $\Delta \leq \chi_i(G) \leq \Delta^2 - \Delta + 1$, where Δ is the maximum degree of G. They characterized the graphs for which the bound is attained in the inequality. They also reported some interesting results on injective colorings of Cartesian graph products, especially of hypercubes. Subsequently, many studies have considered the problem of injective coloring to obtain various results (see [3,5,13–15] for examples).

Graph operations are important methods for constructing new graphs, and they play key roles in the design and analysis of networks, where these graph operations include the Cartesian product, direct product, lexicographic product, union, join, and disjunction of graphs. Their detailed definitions are given in the following sections.

In this study, we consider the injective coloring of graph operations. We also give some sharp bounds (or exact values) of these graph operations. In the second section, we provide some definitions and results related to the graph products, including

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the Cartesian product, direct product, and lexicographic product of graphs. In the third section, we give the bounds of other graph operations. We provide examples of the bounds to show that they are sharp.

2. Results for the graph product

In this section, we give definitions and results for the graph product [1,8], which are used in the following.

The *Cartesian product* $G\Box H$ of two graphs G and H has the vertex set $V(G) \times V(H)$, and $(u, x), (v, y)$ is an edge of $G\Box H$ if $u = v$ and $xy \in E(H)$ or $uv \in E(G)$ and $x = y$, where $u, v \in V(G)$ and $x, y \in V(H)$.

The *direct product* $G \times H$ of two graphs has the vertex set $V(G) \times V(H)$ and two vertices $(u, x), (v, y)$ are adjacent if both $uv \in E(G)$ and $xy \in E(H)$.

The *lexicographic product* $G\circ H$ of G and H is defined as follows. $V(G \circ H) = V(G)\Box V(H)$, two vertices (u, v) and (u', v') are adjacent if and only if either $uu' \in E(G)$ or $u = u'$, $vv' \in E(H)$. Thus, $G\circ H$ is obtained by substituting a copy $H(u)$ of H for every vertex u of G and joining all the vertices of $H(u)$ with all the vertices of $H(u')$ if $uu' \in E(G)$.

In [4], interesting results were given for injective colorings of Cartesian graph products, especially hypercubes , where the results were stated as follows.

Theorem 2.1 ([4]). *If G and H are connected graphs that are both distinct from K_2, then $\chi_i(G\Box H) \le \chi_i(G)\chi_i(H)$.*

Theorem 2.2 ([4]). *Let Q_n be the hypercube of dimension n. Then,*

(1) $\chi_i(Q_n) = n$ *if and only if n is a power of 2.*
(2) $\chi_i(Q_n) \le 2n - 2$.
(3) $\chi_i(Q_{2n+1}) \le 2\chi_i(Q_{n+1})$.
(4) $\chi_i(Q_{2^m-j}) = 2^m$ *for* $0 \le j \le 3$.

Now, we give some examples of graph Cartesian products as well as the exact values of their injective chromatic numbers. Let S be the set of the following graphs: the ladder graph L_m $(P_m\Box P_2)$, the grid graph $G_{m,n}$ $(P_m\Box P_n)$, and the stacked book graph $B_{m,n}$ $(S_{m+1}\Box P_n)$, where C_m is a cycle of length m, S_m is a star of size of $m + 1$, and $m \ge 2$, $n \ge 2$.

Theorem 2.3. *Let G be a graph.*

(1) *If G is a ladder graph, then* $\chi_i(G) = 3$.
(2) *If G is a grid graph, then* $\chi_i(G) = 4$.
(3) *If G is a stacked graph, then*

$$\chi_i(G) = \begin{cases} m+1, & \text{if } n = 2, \\ m+2, & \text{others.} \end{cases}$$

Proof. Since $\chi_i(G) \ge \Delta(G)$, where Δ is the maximum degree, then we give an injective coloring of G by using $\Delta(G)$ colors, and the theorem is proved.

(1) For the ladder graph. Let $G = P_n\Box P_2$, where P_n is a path of length n. Then, $\Delta(G) = 3$. Define the colors as a set $\{1, 2, 3\}$. Now, we give the coloring of G as follows. For the first P_n of G, we color the vertices of P_n as the sequence $a, a, c, b, b, c, a, a, c, \ldots$. For the second P_n of G, we color the corresponding vertices to the first P_n as the sequence $b, b, c, a, a, c, b, b, c, \ldots$, where $a, b, c \in \{1, 2, 3\}$ (see Fig. 1).
We can then confirm that for any vertex v of G: if $deg(v) = 2$, the neighbor is colored by two different colors of $\{a, b, c\}$; if $deg(v) = 3$, the neighbor is colored by three colors. Thus, it is an injective coloring of G.
(2) For the grid graph. Define the color set as $\{1, 2, 3, 4\}$. We give the following colorings. We color G in four rows (if there are least four rows, we color them in the same way) in the following manner: the first row is colored in a sequence as $\{a, a, b, b, a, \ldots\}$, the second row is colored as a sequence as $\{c, c, d, d, c, c, d, d, \ldots\}$, the third row is colored as $\{b, b, a, a, b, b, a, a, \ldots\}$, and the last row is colored as $\{d, d, c, c, d, d, c, c, \ldots\}$, where $a, b, c, d \in \{1, 2, 3, 4\}$ (see Fig. 2).
It is easy to check that this coloring of G is an injective coloring.
(3) For the stacked graph, we give the coloring as follows. Color P_n in graph G using the colors as a sequence $\{m + 1, m + 1, m + 2, m + 2, m + 1, m + 1, m + 2, m + 2, \ldots\}$. We give an order to the leaves of the star. Then, using this order, we color the stars successively by four stars, where the first two stars color the leaves as $\{1, 2, 3, \ldots, m\}$ and the second two stars color the leaves as $\{2, 3, \ldots, m, 1\}$. See Fig. 3. Thus, this is an injective coloring. □

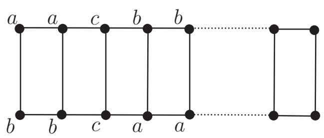

Fig. 1. The injective coloring of G.