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A new class of methods with higher order of convergence for solving systems of nonlinear equations

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ABSTRACT

By studying the commonness of some fifth order methods modified from third order ones for solving systems of nonlinear equations, we propose a new class of three-step methods of convergence order five by modifying a class of two-step methods with cubic convergence. Next, for a given method of order $p \geq 2$ which uses the extended Newton iteration $y_k = x_k - aF'(x_k)^{-1}F(x_k)$ as a predictor, a new method of order $p+2$ is proposed. For example, we construct a class of $m+2$-step methods of convergence order $2m+3$ by introducing only one evaluation of the function to each of the last m steps for any positive integer m. In this paper, we mainly focus on the class of fifth order methods when $m=1$. Computational efficiency in the general form is considered. Several examples for numerical tests are given to show the asymptotic behavior and the computational efficiency of these higher order methods.

1. Introduction

Solving systems of nonlinear equations is an important and interesting problem in numerical analysis and applied scientific branches. For a given nonlinear system $F(x): D \subseteq R^n \to R^n$, the problem is to find a vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)^t$ such that $F(\alpha) = 0$, where

$$F(x) = (f_1(x), f_2(x), \ldots, f_n(x))^t, \quad \text{and} \quad x = (x_{(1)}, x_{(2)}, \ldots, x_{(n)})^t.$$

The best known and most widely used algorithm is the quadratically convergent Newton's method [1,2] defined by

$$x_{k+1} = x_k - F'(x_k)^{-1}F(x_k), \quad k = 0, 1, 2, \ldots, \tag{1.1}$$

where $F'(x)^{-1}$ is the inverse of the first Fréchet derivative $F'(x)$ of the function $F(x)$.

Recently, a number of robust and efficient methods have been proposed to improve the order of convergence of Newton's method in the literature; for example, see [3–15] and references therein. However, the proposed higher order iterative methods are futile unless they have low computational cost. Therefore, the aim in developing new algorithms is to achieve as high as possible convergence order requiring as small as possible the evaluations of functions, derivatives and matrix inversions. We here construct a new class of efficient methods with fifth order convergence based on a class of two-step methods with cubic convergence, and then introduce a technique in which the order of convergence p of a given iterative method is improved to $p+2$.

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This paper is organized as follows. In Section 2, a class of fifth order methods and its extension with higher order of convergence are developed. Computational efficiency in the general form is discussed in Section 3. Several numerical examples are considered in Section 4 to show the asymptotic behavior of these methods. Finally, conclusions are made in Section 5.

2. A new class of efficient methods with higher order of convergence

To start with, we recall some well-known iterative methods for solving systems of nonlinear equations. In [16], Weerakoon and Fernando proposed a two-step method with cubic convergence as follows:

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k), \\ x_{k+1} &= x_k - 2[F'(y_k) + F'(x_k)]^{-1}F(x_k). \end{aligned} \tag{2.1}$$

Note that this scheme has also been developed in [6] independently. Cordero et al. [14] modified it to be a three-step method with fifth order convergence as follows:

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k), \\ z_k &= x_k - 2[F'(y_k) + F'(x_k)]^{-1}F(x_k), \\ x_{k+1} &= z_k - F'(y_k)^{-1}F(z_k). \end{aligned} \tag{2.2}$$

It is obvious that this method requires the evaluations of two functions, two first derivatives and three matrix inversions per iteration.

Homeier [8] developed another two-step method of convergence order three

$$\begin{aligned} y_k &= x_k - \frac{1}{2}F'(x_k)^{-1}F(x_k), \\ x_{k+1} &= x_k - F'(y_k)^{-1}F(x_k). \end{aligned} \tag{2.3}$$

Sharma and Gupta [17] extended it to be a three-step method of convergence order five

$$\begin{aligned} y_k &= x_k - \frac{1}{2}F'(x_k)^{-1}F(x_k), \\ z_k &= x_k - F'(y_k)^{-1}F(x_k), \\ x_{k+1} &= z_k - [2F'(y_k)^{-1} - F'(x_k)^{-1}]F(z_k). \end{aligned} \tag{2.4}$$

While this method requires only two matrix inversions per iteration.

By taking into account the computational cost of (2.2) and (2.4), we wonder whether there exists another fifth order method modified from (2.1) such that the first two steps of this method are the same as (2.2) and the third step can be written in the form of

$$x_{k+1} = z_k - \left\{u\left[F'(y_k) + F'(x_k)\right]^{-1} + vF'(x_k)^{-1}\right\}F(z_k),$$

where u, v are determined constants.

In order to answer this question, we first consider a class of two-step methods of convergence order three given by

$$\begin{aligned} y_k &= x_k - aF'(x_k)^{-1}F(x_k), \\ x_{k+1} &= x_k - [bF'(y_k) + cF'(x_k)]^{-1}F(x_k), \end{aligned} \tag{2.5}$$

where $a \neq 0$ and the constants b, c are to be specified later. We first recall the result of Taylor's expansion on vector functions (see [1]), and then use it to prove our main theorems.

Lemma 1. *Let $F : D \subseteq R^n \to R^n$ be p time Fréchet differentiable in a convex set $D \subseteq R^n$, then for any $x, h \in R^n$, one has that*

$$F(x+h) = F(x) + F'(x)h + \frac{1}{2!}F''(x)h^2 + \cdots + \frac{1}{(p-1)!}F^{(p-1)}(x)h^{p-1} + R_p, \tag{2.6}$$

where

$$\|R_p\| \le \frac{1}{p!}\sup_{0\le t\le 1}\|F^{(p)}(x+th)\|\,\|h\|^p, \quad h^p = (h, h, \overset{p}{\cdots}, h),$$

and $\|M_{kl}\|$ is an arbitrary norm of the matrix M_{kl}.

Theorem 1. *Let $F : D \subseteq R^n \to R^n$ be fourth Fréchet differentiable in a convex set $D \subseteq R^n$ containing the zero α of $F(x)$. Suppose that $F'(x)$ is nonsingular in α. Then, the sequence $\{x_k\}_{k\ge 0}$ ($x_0 \in D$) obtained by construction (2.5) converges to α with order three if and only if $ab = 1/2$ and $b + c = 1$.*

Proof. Since $F(\alpha) = 0$, then Taylor's expansion (2.6) for $F(x_k)$ about α is

$$F(x_k) = F'(\alpha)(x_k - \alpha) + \frac{1}{2!}F''(\alpha)(x_k - \alpha)^2 + \frac{1}{3!}F'''(\alpha)(x_k - \alpha)^3 + O(||x_k - \alpha||^4).$$