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Symmetric solutions for a class of singular biharmonic elliptic systems involving critical exponents ☆

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ABSTRACT

This paper deals with a class of singular biharmonic elliptic systems involving critical exponents in a bounded symmetric domain. By using the variational method and the symmetric criticality principle of Palais, we obtain several existence and multiplicity results of G-symmetric solutions for the systems.

1. Introduction

In this paper, we investigate the existence and multiplicity of nontrivial solutions for the following singular biharmonic elliptic system

$$\begin{cases} \Delta^2 u = K(x)\left(|u|^{2^{**}-2}u + \dfrac{\eta\alpha}{2^{**}}|u|^{\alpha-2}u|v|^{\beta}\right) + \lambda\dfrac{q_1|u|^{q_1-2}u|v|^{q_2}}{(q_1+q_2)|x|^{\varsigma}}, & \text{in } \Omega, \\ \Delta^2 v = K(x)\left(|v|^{2^{**}-2}v + \dfrac{\eta\beta}{2^{**}}|u|^{\alpha}|v|^{\beta-2}v\right) + \lambda\dfrac{q_2|u|^{q_1}|v|^{q_2-2}v}{(q_1+q_2)|x|^{\varsigma}}, & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial n} = 0, \quad v = \dfrac{\partial v}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N > 4)$ is a smooth bounded domain, $0 \in \Omega$ and Ω is G-symmetric with respect to a closed subgroup G of $O(\mathbb{N})$ (see Section 2 for details), $\eta \geq 0$, $\lambda \geq 0$, $0 \leq \varsigma < 4$, $q_1, q_2 > 1$ and $2 < q_1 + q_2 < 2^{**}(\varsigma)$, with $2^{**}(\varsigma) \triangleq \frac{2(N-\varsigma)}{N-4}$, $\alpha, \beta > 1$ satisfy $\alpha + \beta = 2^{**}(0)$, $2^{**}(0) = 2^{**} \triangleq \frac{2N}{N-4}$ is the critical Sobolev exponent, $\frac{\partial}{\partial n}$ is the outer normal derivative, and $K \in C(\overline{\Omega}) \cap L^{\infty}(\overline{\Omega})$ fulfills some symmetry conditions which will be specified later.

The critical growth in singular elliptic problems of second order has been extensively studied in recent years, starting with the seminal paper [1]. The results relating to these problems can be found in [2–4], and the references therein. In a recent paper, Deng and Jin [5] considered the existence of nontrivial solutions of the following critical singular problem

$$-\Delta u = \mu\frac{u}{|x|^2} + K(x)|x|^{-s}u^{2^*(s)-1}, \qquad \text{and} \quad u > 0 \quad \text{in} \quad \mathbb{R}^N, \tag{1.2}$$

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where $\mu \in [0, (\frac{N-2}{2})^2)$, $s \in [0, 2)$, are constants, $N > 2$, $2^*(s) = \frac{2(N-s)}{N-2}$ is the critical Hardy–Sobolev exponent and $2^*(0) = 2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent, and K fulfills certain symmetry conditions with respect to a subgroup G of $O(\mathbb{N})$. By using analytic techniques together with standard variational arguments, the authors proved the existence and multiplicity of G-symmetric solutions to (1.2) under certain hypotheses on K. Subsequently, Waliullah [6] improved the results in [5] by establishing a variant of concentration–compactness principle due to Lions. In particular, Waliullah studied the following elliptic problem

$$(-\Delta)^m u = K(x)|u|^{2^*_{(m)}-2}u \qquad \text{in} \quad \mathbb{R}^N, \tag{1.3}$$

where $m > 1$, $N > 2m$, $2^*_{(m)} = \frac{2N}{N-2m}$ and K is G-symmetric. Using the minimizing sequence and the concentration–compactness principle, the author obtained the existence of nontrivial G-symmetric solution to Eq. (1.3). Very recently, Deng and Huang [7–9] extended the results in [5,6] to the scalar weighted elliptic problems in a bounded G-symmetric domain. Besides, we also mention that when $\mu = s = 0$ and the right-hand side term $|x|^{-s}u^{2^*(s)-1}$ is replaced by u^{q-1} $(1 < q < \frac{2N}{N-2}$ or $q = \frac{2N}{N-2})$ in (1.2), some elegant results of G-symmetric solutions of (1.2) were established in [10–12]. Finally, when $G = O(\mathbb{N})$, we remark that Su and Wang [13] proved the existence of nontrivial radial solutions for a class of quasilinear singular equations such as (1.2) with radial potentials by establishing several new embedding theorems.

On the other hand, there have been many papers concerned with the existence and multiplicity of nontrivial solutions for second-order semilinear elliptic systems. In [14], Wu considered the following elliptic system with subcritical nonlinearity and sign-changing weights

$$\begin{cases} -\Delta u = \dfrac{2\alpha}{\alpha+\beta}K(x)|u|^{\alpha-2}u|v|^{\beta} + \lambda f(x)|u|^{q-2}u, & \text{in } \Omega, \\ -\Delta v = \dfrac{2\beta}{\alpha+\beta}K(x)|u|^{\alpha}|v|^{\beta-2}v + \sigma h(x)|v|^{q-2}v, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $1 < q < 2$, $\alpha, \beta > 1$ satisfy $2 < \alpha + \beta < 2^*$ and the weight function K, f, h fulfills certain suitable conditions. Applying the analytic techniques of Nehari manifold, the author proved that the system (1.4) had at least two nonnegative solutions if the parameters λ and σ satisfied an appropriate condition. Subsequently, Nyamoradi [15] generalized some results of [14] to the singular elliptic systems involving critical Hardy–Sobolev exponents. Very recently, Huang and Kang [16] studied the existence and asymptotic properties of positive solutions of the following critical singular elliptic systems

$$\begin{cases} -\Delta u - \mu_1 \dfrac{u}{|x-a_1|^2} = |u|^{2^*-2}u + \dfrac{\eta\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta} + \lambda_1|u|^{q_1-2}u, & \text{in } \Omega, \\ -\Delta v - \mu_2 \dfrac{v}{|x-a_2|^2} = |v|^{2^*-2}v + \dfrac{\eta\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v + \lambda_2|v|^{q_2-2}v, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $\eta > 0$, $a_i \in \Omega$, $\lambda_i > 0$, $\mu_i < (\frac{N-2}{2})^2$, $2 \le q_i < 2^*(i = 1, 2)$, and $\alpha, \beta > 1$ satisfy $\alpha + \beta = 2^*$. Note that $|u|^{\alpha-2}u|v|^{\beta}$ and $|u|^{\alpha}|v|^{\beta-2}v$ in (1.5) are called strongly coupled terms, and $|u|^{2^*-2}u$, $|v|^{2^*-2}v$ are weakly coupled terms. By employing the Moser iteration method and variational methods, the authors obtained the existence of positive solutions and some properties of the nontrivial solutions to (1.5). Other results about existence and multiplicity of nontrivial solutions, also for second-order elliptic systems, can be found in [17–19]. For the systems of fourth-order elliptic equations with concave–convex nonlinearities, various studies concerning the solutions structures have also been studied by several authors, see [20] and references therein.

However, concerning the existence and multiplicity of G-symmetric solutions for elliptic systems, we can only find some existence results for singular second-order elliptic systems in [21,22] and when $G = O(\mathbb{N})$, some radial and nonradial results for nonsingular second-order elliptic systems in [23]. Stimulated by [5,10,16], in the present paper, we shall study the existence and multiplicity of G-symmetric solutions for the fourth-order elliptic system (1.1). Due to the singular perturbations $|x|^{-\varsigma}|u|^{q_1-2}u|v|^{q_2}$, $|x|^{-\varsigma}|u|^{q_1}|v|^{q_2-2}v$, four nonlinear strongly coupled terms $|u|^{\alpha-2}u|v|^{\beta}$, $|u|^{\alpha}|v|^{\beta-2}v$ and weakly coupled terms $|u|^{2^{**}-2}u$ and $|v|^{2^{**}-2}v$ with the critical Sobolev exponents, compared with (1.2) and (1.3), the singular biharmonic elliptic system (1.1) becomes more complicated to deal with and therefore we have to face more difficulties. As far as we know, it seems that there are no results for (1.1) even in the scalar case $\eta = \lambda = 0$ and $u = v$. Many attractive and challenging topics on singular fourth-order elliptic systems remain unsolved. Hence, it makes sense for us to investigate system (1.1) thoroughly. Let $K_0 > 0$ be a constant. Note that here we will try to treat both the cases of $\lambda = 0$, $K(x) \not\equiv K_0$, and $\lambda > 0$, $K(x) \equiv K_0$.

This paper is organized as follows. In Section 2, we will establish the suitable Sobolev space which is useful to discuss the elliptic system (1.1), and we will state the main results of this paper. In Section 3, we detail the proofs of several existence and multiplicity results for the cases $\lambda = 0$ and $K(x) \not\equiv K_0$ in (1.1). In Section 4, we present the proofs of existence results for the cases $\lambda > 0$ and $K(x) \equiv K_0$ in (1.1). Our methods in this paper are mainly based upon the symmetric criticality principle of Palais (see [24]) and variational arguments.

2. Preliminaries and main results

Let $O(\mathbb{N})$ be the group of orthogonal linear transformations of $\mathbb{R}^N$ with natural action and let $G \subset O(\mathbb{N})$ be a closed subgroup. For any point $x \in \mathbb{R}^N$, the set $G_x = \{x_1 \in \mathbb{R}^N; x_1 = gx, g \in G\}$ is called an orbit of x. The number of points contained in the orbit G_x