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Robust a posteriori error estimates for conforming and nonconforming finite element methods for convection–diffusion problems ☆

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ABSTRACT

A posteriori error estimation is carried out within a unified framework for various conforming and nonconforming finite element methods for convection–diffusion problems. Our main contribution is finding an appropriate norm to measure the error, which incorporates a discrete energy norm, a discrete dual semi-norm of the convective derivative and jumps of the approximate solution over element faces (edges in two dimensions). The error estimator is shown to be robust with respect to the Péclet number in the sense of the modified norm. Based on a general error decomposition, we show that the key ingredient of error estimation is the estimation on the consistency error related to the particular numerical scheme, and the remaining terms can be bounded in a unified way. The numerical results are presented to illustrate the robustness and practical performance of the estimator in an adaptive refinement strategy.

1. Introduction

The convection–diffusion problem arises in a vast number of applications. The solution may display thin internal or boundary layers in the convection-dominated case, such that the standard Galerkin finite element method has poor stability and accuracy properties. To achieve stability and accuracy simultaneously, a wide range of stabilized finite element methods have been proposed, such as streamline-diffusion methods, local projection schemes, subgrid viscosity methods and continuous interior penalty methods, see [28,32] and the references therein for an overview. A posteriori error analysis of streamline-diffusion methods is already well-understood owing to the pioneering work of Verfürth [33,34]. Specifically, two types of estimators can be derived, i.e., semi-robust and robust error estimators. We refer the readers to the review book [35] for the details. Here we focus on the robust error estimation where the constant factors are independent of the Péclet number, but the error norm has to be modified and included the convective derivative of the error. In [34], the convective derivative is measured in a dual norm (also see [29] for a different norm). It is worth mentioning the latest work [32] of Tobiska and Verfürth, where the robust error estimation is extended to other stabilized finite element methods. There are also other proposals in the literature for estimating the error with respect to a priori given (sometimes mesh-dependent) norm, such as residual error indicators [3,23], hierarchical estimates [1], averaging techniques [10] or functional (constant-free) error estimates [13]. All the works mentioned above are

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only limited to the conforming methods where the discrete solution space is in the primary space. Finally, we also mention that the a posteriori error estimates for the mixed finite element method have been developed, see, e.g. [11,12,36].

Recently, a great interest arises for a better understanding of the convergence properties of the nonconforming finite elements for the convection–diffusion problem. A class of nonconforming streamline-diffusion finite element methods have been proposed and analyzed in [21,22]. Therein, theoretical and numerical investigations show that by adding certain jump terms to the bilinear form the known $\mathcal{O}(h^{3/2})$ order for L^2-convergence in the streamline diffusion norm can be preserved as in the conforming case for piecewise linear polynomials. We refer the readers to [25–27,31] for further survey and analysis on the nonconforming streamline-diffusion methods. In [14], El Alaoui and Ern designed and analyzed a nonconforming finite element method with subgrid viscosity to solve the convection–diffusion problem, which provided the first extension of the subgrid viscosity technique introduced by Guermond [18,19] to nonconforming settings. The nonconforming finite element methods with face and interior penalty were discussed in [4,15].

Compared with the conforming elements, a posteriori error analysis of nonconforming finite element approximations to convection–diffusion equations is a much less explored topic. El Alaoui et al. [15] derived a semi-robust error estimator for the nonconforming finite element discretization with face penalty. In our latest work [38], semi-robust a posteriori error estimation for the nonconforming streamline-diffusion methods [22] was presented, and the error analysis was also shown to be applied to other nonconforming finite element methods with face penalty [15] and subgrid viscosity [14]. This can be viewed as a first step towards establishing robust error estimates in the nonconforming setting. Besides, for the interior penalty discontinuous Galerkin method the semi-robust error estimates have also been derived by Ern et al. [16,17].

However, it is not direct to extend the robust error estimates to the nonconforming finite element discretizations, since the finite element space is nonconforming, i.e., not in the primary space, which requires an appropriate norm for the nonconforming space. Specifically, when considering the robust error estimation for the nonconforming finite element methods in the above modified norm, one will meet two obstacles. The first one is that the dual norm introduced in [34] is not well-defined in the nonconforming finite element space, which needs to redefine the dual norm appropriate for the functions of the nonconforming space. The second one is that the norm has to contain an extra contribution from jumps of the nonconforming approximation to bound the tangential jump terms in a posteriori error estimation, since the jumps over element faces (edges in two dimensions) do not vanish for the nonconforming approximation, see [38, Section 6] for the detailed discussion.

In this paper, we should overcome the two obstacles mentioned above, and extend the robust error estimates to the nonconforming cases. In Section 3, we first introduce an appropriate norm to measure the error for the conforming and nonconforming methods, which incorporates a discrete energy norm, a discrete dual semi-norm of the convective derivative, and jumps of the approximate solution over element faces (edges in two dimensions) with weights dependent on the mesh size. For the conforming approximation, the jump terms vanish and the present norm become the usual one [34]. The jump terms with different weights can be found in the definitions of the error norm for the discontinuous Galerkin methods [30,39] and hybridizable discontinuous Galerkin method [7] in order to derive a robust a posteriori error estimator. Specifically, the error norm in [30,39] contains the jump term $(h_F\varepsilon^{-1}\|[\![u_h]\!]\|_F^2)^{1/2}$ on each interior interface of meshes, and the one in [7] contains a jump term $(\gamma_F\|[\![u_h]\!]\|_F^2)^{1/2}$ instead where for the definition of weight γ_F we refer to [7, (2.7)]. In contrast, the error norm in this paper contains the jump term $(h_F^{-1}\alpha_F^2\|[\![u_h]\!]\|_F^2)^{1/2}$ with $\alpha_F = \min\{\varepsilon^{-1/2}h_F, 1\}$, We note that the weight $h_F^{-1}\alpha_F^2$ is comparable with the weight γ_F in [7]. Therefore, as well as the one in [7] our a posteriori error estimator will not enlarge the error estimate too much as the error estimator in [30,39], when the mesh size is not small compared with the diffusion coefficient.

And then, an abstract framework is presented for arbitrary conforming and nonconforming finite element discretizations of the convection–diffusion problem. Therein, the error is decomposed into three terms: residual error, consistency error and nonconforming error, which is fixed for all the conforming and nonconforming approximations. Note that, for the conforming approximations the nonconforming error vanishes and the whole error analysis is the same to the case in [32]. The residual error and nonconforming error can be estimated in a unified way, as shown in Lemmas 3.1 and 3.2. Thus, the remaining work is the estimation on the consistency error related to the particular discretization, see Theorem 3.1. In addition, the lower error bound is obtained without a notion of any numerical scheme, see Theorem 3.2.

In Section 4, we briefly discuss the estimation on the consistency error for various conforming finite element methods, such as streamline-diffusion methods, local projection schemes, subgrid viscosity methods and continuous interior penalty methods. In Section 5 we present the estimates of the consistency error for various nonconforming finite element methods, such as nonconforming streamline-diffusion methods (convective and skew-symmetrized forms), nonconforming face penalty method and nonconforming subgrid viscosity method. For other nonconforming methods, such as the P_1^{mod} streamline-diffusion method [25] and nonconforming interior penalty [4], the case is similar. In the final section, we show the numerical results for two examples with internal and boundary layers solved by the nonconforming streamline-diffusion methods.

2. Notations and preliminaries

Let $\Omega \subseteq \mathbb{R}^d$ $(d = 2, 3)$, be a bounded domain with polygonal or polyhedral boundary $\partial\Omega$. For any given open subset S of Ω, $(\cdot,\cdot)_S$ and $\|\cdot\|_S$ denote the usual integral inner product and the corresponding norm of both $L^2(S)$ and $[L^2(S)]^d$, respectively. If $S = \Omega$, the subscript will be omitted.

We consider the convection–diffusion problem with homogeneous boundary condition

$$-\varepsilon\Delta u + \mathbf{b}\cdot\nabla u + cu = f, \text{ in } \Omega, \quad u = 0, \text{ on } \partial\Omega, \tag{2.1}$$