Applied Mathematics and Computation 264 (2015) 378–391

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Time-dependent Hermite–Galerkin spectral method and its applications

Xue Luo [a], Shing-Tung Yau [b], Stephen S.-T. Yau [c,*]

[a] School of Mathematics and System Sciences, Beihang University, Beijing 100191, PR China
[b] Department of Mathematics, Harvard University, Cambridge, MA 02138, USA
[c] Department of Mathematical Sciences, Tsinghua University, Beijing 100084, PR China

ARTICLE INFO

MSC:
65N35
65N22
65M70

Keywords:
Hermite–Galerkin spectral method
Time-dependent parameters
Nonlinear convection–diffusion equations

ABSTRACT

A time-dependent Hermite–Galerkin spectral method (THGSM) is investigated in this paper for the nonlinear convection–diffusion equations in the unbounded domains. The time-dependent scaling factor and translating factor are introduced in the definition of the generalized Hermite functions (GHF). As a consequence, the THGSM based on these GHF has many advantages, not only in theoretical proofs, but also in numerical implementations. The stability and spectral convergence of our proposed method have been established in this paper. The Korteweg–de Vries–Burgers (KdVB) equation and its special cases, including the heat equation and the Burgers' equation, as the examples, have been numerically solved by our method. The numerical results are presented, and it surpasses the existing methods in accuracy. Our theoretical proof of the spectral convergence has been supported by the numerical results.

1. Introduction

Many scientific and engineering problems are naturally modeled in the unbounded domains. One way to numerically solve the problems is to restrict the model equation in some bounded domain and artificially impose some boundary condition cleverly. Whereas this introduces errors even before the implementation of the numerical scheme. Another more suitable way is to use the spectral approaches employing orthogonal systems in unbounded domain, such as using Laguerre polynomials for the problems in semi-bounded or exterior domains [13,19], and using Hermite polynomials for those in the whole space [1,10,11,15,22].

Although the freedom from artificial boundary condition is very attractive, the Hermite spectral method (HSM) is only widely studied in the recent decade, due to its poor resolution without the appropriate scaling factor. The Hermite functions, defined as $\{H_n(x)e^{-x^2}\}_{n=0}^{\infty}$, have the same deficiency as the polynomials $\{H_n(x)\}_{n=0}^{\infty}$. However, a suitably chosen scaling factor will greatly improve the resolution. Its importance has been discussed in [25,28]. It has been shown in [4] that the scaling factor should be selected according to the truncated modes N and the asymptotical behavior of the function $f(x)$, as $|x| \to \pm\infty$. The optimal scaling factor is still an open problem, even in the case that $f(x)$ is given explicitly, to say nothing of the exact solution to a differential equation, which is in general unknown beforehand. Recently, during the study of using the HSM to solve the nonlinear filtering problems, the first and the third author of this paper gave a practical strategy in [21] to pick the appropriate scaling factor and the corresponding truncated mode for at least the most commonly used types of functions, i.e. the Gaussian type and the super-Gaussian type. Thanks to this guideline, the Hermite–Galerkin spectral method (HGSM) becomes implementable.

* Corresponding author. Tel.: +86 10 62787874.
E-mail addresses: xluo@buaa.edu.cn (X. Luo), yau@math.harvard.edu (S.-T. Yau), yau@uic.edu (S.S.-T. Yau).

http://dx.doi.org/10.1016/j.amc.2015.04.088
0096-3003/

In the literature of solving partial differential equations in unbounded domains using HSM, nearly all the schemes are not direct Galerkin ones. As far as the author knows, there are at least two possible reasons:

1. The lack of the practical guidelines of choosing appropriate scaling factor makes the direct Galerkin method infeasible.
2. When directly applying traditional definition of Hermite functions (i.e., $\{H_n(x)e^{-x^2}\}_{n=0}^{\infty}$) to second-order differential equations, it is found in [12] that the stiff matrix is of nonsymmetric bilinear form, which has no property of coerciveness. In other word, the stability cannot be established by using the classical energy method.

To overcome the obstacles above, Funaro and Kavian [11] first consider the use of the Hermite polynomials to approximate the solutions of some diffusion evolution equations in unbounded domains. The variable transformation technique is introduced to get better resolution. Later, Guo [12], Guo and Xu [15] developed the Hermite polynomial spectral and pseudo-spectral methods, where the transformation $U = e^{-x^2}V$ is used, and then V is approximated by the Hermite polynomials. Ma et al. [22], Ma and Zhao [23] introduced a time-dependent parameter to stabilize the scheme, which is based on the traditional defined Hermite functions. However, no discussion was given on how to choose such parameter for the particular problems.

The aim of this paper is to develop a time-dependent Hermite–Galerkin spectral method (THGSM) to approximate the solution to the nonlinear convection–diffusion equations with high accuracy. In this paper, we focus on the nonlinearity satisfying *Assumptions 1 and 2* in Section 3, which includes the Burger's equation and KdVB equation. The spectral method has been applied to various nonlinear problems, such as the nonlinear Shröndinger equation [2], Bose–Einstein condensates [3], Navier–Stokes equation [20] etc. Also different types of differential equations have been studied by using spectral method, for instance, pantography-type differential and integral equations [5], hyperbolic PDEs [6], pattern-forming nonlinear evolution systems [7] etc. More thorough discussion of spectral methods is referred to [26]. The time-dependence is reflected in the definition of the generalized Hermite functions (GHF), where the scaling factor and the translating factor are the functions of time. The choice of the time-dependent scaling factor can follow the guidelines in [21], while the time-dependent translating factor mainly deals with the time-shifting of the solution, see examples in Section 4.3. The advantages of our THGSM are the following:

1. It is a direct Galerkin scheme, which can be implemented straightforward. And the resulting stiffness matrix of the second-order differential equations is of nice properties. For example, it is tri-diagonal, symmetric and diagonally dominant in the linear case, i.e. $g(u) \equiv 0$ in (3.1); it is symmetric for the Burgers' equation, i.e. $g(u) = \frac{u}{2}$ in (3.1).
2. The proofs of stability and spectral convergence are greatly simplified, thanks to the definition of GHF. They are analyzed in the L^2 space, instead of the weighted one as in [22].
3. From the numerical simulations in Section 4, our scheme outperforms nearly all the existing methods in accuracy.

An outline of the paper is as follows. In Section 2, we give the definition of GHF and its properties. For the readers' convenience, we include the proof of the error estimate of the orthogonal projection. Our TGHSM to solve the nonlinear convection–diffusion equations is introduced in Section 3. The stability analysis in the sense of [12] and the spectral convergence are shown there. Section 4 is devoted to the numerical simulations, where we compared the numerical results with those obtained by other methods in some benchmark equations.

2. Generalized Hermite functions (GHF)

We introduce the GHF and derive some properties which are inherited from the physical Hermite polynomials. For the sake of completeness, we give the proof of the convergence rate of the orthogonal approximation.

2.1. Notations and preliminaries

Let $L^2(\mathbb{R})$ be the Lebesgue space, which equips with the norm $\|\cdot\| = (\int_{\mathbb{R}} |\cdot|^2 dx)^{\frac{1}{2}}$ and the scalar product $\langle \cdot,\cdot \rangle$.

Let $\mathcal{H}_n(x)$ be the physical Hermite polynomials given by $\mathcal{H}_n(x) = (-1)^n e^{x^2} \partial_x^n e^{-x^2}$, $n \geq 0$. The three-term recurrence

$$\mathcal{H}_0 \equiv 1; \quad \mathcal{H}_1(x) = 2x; \quad \text{and} \quad \mathcal{H}_{n+1}(x) = 2x\mathcal{H}_n(x) - 2n\mathcal{H}_{n-1}(x). \tag{2.1}$$

is more handy in implementation. One of the well-known and useful fact of Hermite polynomials is that they are mutually orthogonal with the weight $w(x) = e^{-x^2}$. We define the time-dependent GHF as

$$H_n^{\alpha,\beta}(x,t) = \left(\frac{\alpha(t)}{2^n n! \sqrt{\pi}}\right)^{\frac{1}{2}} \mathcal{H}_n[\alpha(t)(x-\beta(t))] e^{-\frac{1}{2}\alpha^2(t)[x-\beta(t)]^2}, \tag{2.2}$$

for $n \geq 0$, where $\alpha(t) > 0$, $\beta(t)$, for $t \in [0, T]$, are functions of time. For the conciseness of notation, let us denote $d(n) = \sqrt{\frac{n}{2}}$. And if no confusion will arise, in the sequel we omit the t in $\alpha(t)$ and $\beta(t)$. It is readily to derive the following properties for the GHF (2.2):

- At each time $t > 0$, $\{H_n^{\alpha,\beta}(\cdot,t)\}_{n\in\mathbb{Z}^+}$ form the orthogonal basis of $L^2(\mathbb{R})$, i.e.

$$\int_{\mathbb{R}} H_n^{\alpha,\beta}(x,t) H_m^{\alpha,\beta}(x,t) dx = \delta_{nm}, \tag{2.3}$$

  where δ_{nm} is the Kronecker function.