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Some approximation results by (p, q)-analogue of Bernstein–Stancu operators

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ABSTRACT

In this paper, we introduce a new analogue of Bernstein–Stancu operators based on (p, q)-integers which we call as (p, q)-Bernstein–Stancu operators. We study approximation properties for these operators based on Korovkin's type approximation theorem and also study some direct theorems. Furthermore, we give comparisons and some illustrative graphics for the convergence of operators to some function.

1. Introduction and preliminaries

During the last two decades, the applications of q-calculus emerged as a new area in the field of approximation theory. The rapid development of q-calculus has led to the discovery of various generalizations of Bernstein polynomials involving q-integers. The aim of these generalizations is to provide appropriate and powerful tools to application areas such as numerical analysis, computer-aided geometric design and solutions of differential equations.

In 1987, Lupaş [17] introduced the first q-analogue of Bernstein operators [6] and investigated its approximating and shape-preserving properties. Another q-generalization of the classical Bernstein polynomials is due to Phillips [26]. Several generalizations of well-known positive linear operators based on q-integers were introduced and their approximation properties have been studied by several authors. For instance, q-Baskakov–Kantorovich operators in [11]; q -Szász–Mirakjan operators in [25]; q-Bleimann, Butzer and Hahn operators in [3] and [9]; q-analogue of Baskakov and Baskakov–Kantorovich operators in [18]; q-analogue of Szász–Kantorovich operators in [19]; q-analogue of Stancu–Beta operators in [4] and [21]; and q-Lagrange polynomials in [23] were defined and their approximation properties were investigated.

Recently, Mursaleen et al. introduced and studied approximation properties for new positive linear operators of Lagrange type in [22] and also studied approximation properties of the q-analogue of generalized Berstein–Shurer operators in [20].

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The q-analogue of Bernstein–Stancu operators [24] is given by

$$S_{n,q}(f;x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1} (1-q^s x) f\left(\frac{[k]_q+\alpha}{[n]_q+\beta}\right), x \in [0,1] \tag{1.1}$$

where $S_{n,q}: C[0,1] \to C[0,1]$ are defined for any $n \in \mathbb{N}$ and for any function $f \in C[0,1]$.

Details on the q-calculus can be found in [14] and for the applications of q-calculus in approximation theory, one can refer [5].

In this paper, we introduce a new generalization of q -Bernstein–Stancu operators by using the notion of (p, q)-calculus and call it as (p, q)-Bernstein–Stancu operators. We study the approximation properties based on Korovkin's type approximation theorem and also establish some direct theorems. Furthermore, we show comparisons and some illustrative graphics for the convergence of operators to a function.

Let us recall certain notations of (p, q)-calculus:

The (p, q)-integers were introduced in order to generalize or unify several forms of q-oscillator algebras well known in the earlier physics literature related to the representation theory of single parameter quantum algebras [7]. The (p, q)-integers $[n]_{p,q}$ are defined by

$$[n]_{p,q} := \frac{p^n - q^n}{p-q}, \quad n = 0, 1, 2, \dots, \quad 0<q<p \le 1.$$

The (p, q)-binomial expansion is

$$(ax+by)^n_{p,q} := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} a^{n-k} b^k x^{n-k} y^k$$

$$(x+y)^n_{p,q} := (x+y)(px+qy)(p^2x+q^2y)\cdots(p^{n-1}x+q^{n-1}y).$$

Also, the (p, q)-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} := \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.$$

Details on (p, q)-calculus can be found in [12,13,15,27]. For $p = 1$, all the notions of (p, q)-calculus are reduced to q-calculus.

2. Construction of operators

Motivated by the idea of (p, q)-calculus and its importance, we introduce (p, q)-analogue of Bernstein–Stancu operators as follows: for $0 < q < p \le 1$ and $n \in \mathbb{N}$,

$$S_{n,p,q}(f;x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x) f\left(\frac{[k]_{p,q}+\alpha}{[n]_{p,q}+\beta}\right), \quad x \in [0,1] \tag{2.1}$$

First, we prove the following basic lemmas:

Lemma 2.1. *For $x \in [0, 1]$, $0 < q < p \le 1$*

(i) $S_{n,p,q}(1;x) = 1$;

(ii) $S_{n,p,q}(t;x) = \frac{[n]_{p,q}x}{[n]_{p,q}+\beta} + \frac{\alpha}{[n]_{p,q}+\beta}$;

(iii) $S_{n,p,q}(t^2;x) = \frac{1}{[n]_{p,q}+\beta^2}[[n]_{p,q}(xp+1-x)^{n-1}x + q[n]_{p,q}[n-1]_{p,q}x^2 + 2\alpha[n]_{p,q}x + \alpha^2]$.

For typographical convenience, throughout the paper, we write $(px+1-x)^{n-1}_{p,q}$ for $(px+1-x)^{n-1}$.

Proof.

(i)

$$S_{n,p,q}(1;x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x) = 1.$$

(ii)

$$\begin{aligned} S_{n,p,q}(t;x) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x) \frac{[k]_{p,q}+\alpha}{[n]_{p,q}+\beta} \\ &= \frac{1}{[n]_{p,q}+\beta}\left[[n]_{p,q} \sum_{k=1}^{n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x) + \alpha\right] \\ &= \frac{[n]_{p,q}x}{[n]_{p,q}+\beta} + \frac{\alpha}{[n]_{p,q}+\beta}. \end{aligned}$$