Applied Mathematics and Computation 264 (2015) 453–465

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Determinantal representations of the W-weighted Drazin inverse over the quaternion skew field

Ivan Kyrchei*

Pidstrygach Institute for Applied Problems of Mechanics and Mathematics of NAS of Ukraine, str. Naukova 3b, Lviv 79060, Ukraine

ARTICLE INFO

MSC:
15A09
15B33

Keywords:
W-weighted Drazin inverse
Quaternion matrix
Column determinant
Row determinant

ABSTRACT

Within the framework of the theory of the column and row determinants, we obtain new determinantal representations of the W-weighted Drazin inverse over the quaternion skew field. We give determinantal representations of the W-weighted Drazin inverse by using previously introduced determinantal representations of the Drazin inverse, the Moore–Penrose inverse, and the limit representations of the W-weighted Drazin inverse in some special case.

1. Introduction

Let $\mathbb{R}$ and $\mathbb{C}$ be the real and complex number fields, respectively. Throughout the paper, we denote the set of all $m \times n$ matrices over the quaternion algebra

$$\mathbb{H} = \{a_0 + a_1 i + a_2 j + a_3 k \,|\, i^2 = j^2 = k^2 = -1,\ a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

by $\mathbb{H}^{m\times n}$, and by $\mathbb{H}_r^{m\times n}$ the set of all $m \times n$ matrices over $\mathbb{H}$ with a rank r. Let $\mathrm{M}(n, \mathbb{H})$ be the ring of $n \times n$ quaternion matrices and $\mathbf{I}$ be the identity matrix with the appropriate size. For $\mathbf{A} \in \mathbb{H}^{n\times m}$, we denote by $\mathbf{A}^*$, rank $\mathbf{A}$ the conjugate transpose (Hermitian adjoint) matrix and the rank of $\mathbf{A}$. The matrix $\mathbf{A} = (a_{ij}) \in \mathbb{H}^{n\times n}$ is Hermitian if $\mathbf{A}^* = \mathbf{A}$.

The definitions of the generalized inverse matrices may be extended to quaternion matrices.

The Moore–Penrose inverse of $\mathbf{A} \in \mathbb{H}^{m\times n}$, denoted by $\mathbf{A}^\dagger$, is the unique matrix $\mathbf{X} \in \mathbb{H}^{n\times m}$ satisfying the following equations

$$\mathbf{AXA} = \mathbf{A}; \tag{1}$$

$$\mathbf{XAX} = \mathbf{X}; \tag{2}$$

$$(\mathbf{AX})^* = \mathbf{AX}; \tag{3}$$

$$(\mathbf{XA})^* = \mathbf{XA}. \tag{4}$$

For $\mathbf{A} \in \mathbb{H}^{n\times n}$ with $k = Ind\mathbf{A}$ the smallest positive number such that $\operatorname{rank}\mathbf{A}^{k+1} = \operatorname{rank}\mathbf{A}^k$, the Drazin inverse of $\mathbf{A}$, denoted by $\mathbf{A}^D$, is defined to be the unique matrix $\mathbf{X}$ that satisfying (2) and the equations

$$\mathbf{AX} = \mathbf{AX}; \tag{5}$$

$$\mathbf{A}^{k+1}\mathbf{X} = \mathbf{A}^k. \tag{6}$$

In particular, when $Ind\mathbf{A} = 1$, then the matrix $\mathbf{X}$ is called the group inverse of $\mathbf{A}$ and is denoted by $\mathbf{X} = \mathbf{A}^g$.

* Tel.: +380975725731; fax: +380322636270.
E-mail address: kyrchei@lms.lviv.ua, kyrchei@online.ua

http://dx.doi.org/10.1016/j.amc.2015.04.125

If $Ind\mathbf{A} = 0$, then $\mathbf{A}$ is nonsingular, and $\mathbf{A}^D \equiv \mathbf{A}\dagger = \mathbf{A}^{-1}$.

Cline and Greville [1] extended the Drazin inverse of square matrix to rectangular matrix, which can be generalized to the quaternion algebra as follows. For $\mathbf{A} \in \mathbb{H}^{m\times n}$ and $\mathbf{W} \in \mathbb{H}^{n\times m}$, the W-weighted Drazin inverse of $\mathbf{A}$ with respect to $\mathbf{W}$ is the unique solution to the following equations

$$(\mathbf{AW})^{k+1}\mathbf{XW} = (\mathbf{AW})^k; \tag{7}$$

$$\mathbf{XWAWX} = \mathbf{X}; \tag{8}$$

$$\mathbf{AWX} = \mathbf{XWA}, \tag{9}$$

where $k = \max\{Ind(\mathbf{AW}), Ind(\mathbf{WA})\}$. It is denoted by $\mathbf{X} = \mathbf{A}_{d,\mathbf{W}}$. The properties of the W-weighted Drazin inverse for matrices with complex entries can be found in [1–6]. These properties can be generalized to $\mathbb{H}$. If $\mathbf{A} \in \mathbb{H}^{m\times n}$ with respect to $\mathbf{W} \in \mathbb{H}^{n\times m}$ and $k = \max\{Ind(\mathbf{AW}), Ind(\mathbf{WA})\}$, then

$$\mathbf{A}_{d,\mathbf{W}} = \mathbf{A}((\mathbf{WA})^D)^2 = ((\mathbf{AW})^D)^2\mathbf{A}, \tag{10}$$

$$\mathbf{A}_{d,\mathbf{W}}\mathbf{W} = (\mathbf{WA})^D, \mathbf{WA}_{d,\mathbf{W}} = (\mathbf{AW})^D. \tag{11}$$

The problem of determinantal representation of generalized inverse matrices only recently begun to be decided through the theory of the column-row determinants introduced in [7,8].

The theory of row and column determinants develops the classical approach to a definition of a determinant, as alternating sum of products of the entries of matrix but with a predetermined order of factors in each terms of the determinant. A determinant of a quadratic matrix with noncommutative elements is often called the noncommutative determinant. Unlike other known noncommutative determinants such as determinants of Dieudonné [9], Study [10], Moore [11,12], Chen [13], quasideterminants of Gelfand-Retakh [14], the double determinant built on the theory of the column-row determinants has properties similar to a usual determinant, in particular it can be expand along arbitrary rows and columns. This property is necessary for determinantal representations of inverse and generalized inverse matrices. Determinantal representations of the Moore–Penrose inverse and the Drazin inverse over the quaternion skew-field have been obtained in [15] and [16], respectively. Determinantal representations of an outer inverse $\mathbf{A}^{(2)}_{T,S}$ is introduced in [17,18] using the column-row determinants as well. Recall that an outer inverse of a matrix $\mathbf{A}$ over complex field with prescribed range space T and null space S is a solution of (2) with restrictions,

$$\mathcal{R}(\mathbf{X}) = T, \quad \mathcal{N}(\mathbf{X}) = S.$$

Within the framework of the theory of the column-row determinants Song [19] also gave a determinantal representation of the W-weighted Drazin inverse over the quaternion skew-field using its characterization by an outer inverse $\mathbf{A}^{(2)}_{T,S}$. But in obtaining of this determinantal representation have been used auxiliary matrices which different from $\mathbf{A}$ or its powers. In this paper we obtain determinantal representations of the W-weighted Drazin inverse of $\mathbf{A} \in \mathbb{H}^{m\times n}$ with respect to $\mathbf{W} \in \mathbb{H}^{n\times m}$ by using only their entries.

The paper is organized as follows. We start with some basic concepts and results from the theory of the row and column determinants and give the determinantal representations of the inverse, the Moore–Penrose inverse, and the Drazin inverse over the quaternion skew field in Section 2. In Section 3, we obtain determinantal representations of the W-weighted Drazin inverse by using introduced above determinantal representations of the Drazin inverse, the Moore–Penrose inverse, and the limit representations of the W-weighted Drazin inverse in some special case. In Section 4, we show a numerical example to illustrate the main result.

2. Elements of the theory of the column and row determinants

For a quadratic matrix $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ we define n row determinants and n column determinants as follows. Suppose S_n is the symmetric group on the set $I_n = \{1, \ldots, n\}$.

Definition 2.1. The ith row determinant of $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ is defined for all $i = \overline{1, n}$ by putting

$$\begin{aligned}\mathrm{rdet}_i\mathbf{A} &= \sum_{\sigma\in S_n}(-1)^{n-r}a_{ii_{k_1}}a_{i_{k_1}i_{k_1+1}}\ldots a_{i_{k_1+l_1}i}\ldots a_{i_{k_r}i_{k_r+1}}\ldots a_{i_{k_r+l_r}i_{k_r}},\\ \sigma &= \left(i\,i_{k_1}i_{k_1+1}\ldots i_{k_1+l_1}\right)\left(i_{k_2}i_{k_2+1}\ldots i_{k_2+l_2}\right)\ldots\left(i_{k_r}i_{k_r+1}\ldots i_{k_r+l_r}\right),\end{aligned}$$

with conditions $i_{k_2} < i_{k_3} < \cdots < i_{k_r}$ and $i_{k_t} < i_{k_t+s}$ for $t = \overline{2, r}$ and $s = \overline{1, l_t}$.

Definition 2.2. The jth column determinant of $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ is defined for all $j = \overline{1, n}$ by putting

$$\begin{aligned}\mathrm{cdet}_j\,\mathbf{A} &= \sum_{\tau\in S_n}(-1)^{n-r}a_{j_{k_r}j_{k_r+l_r}}\ldots a_{j_{k_r+1}i_{k_r}}\ldots a_{j\,j_{k_1+l_1}}\ldots a_{j_{k_1+1}j_{k_1}}a_{j_{k_1}j},\\ \tau &= \left(j_{k_r+l_r}\ldots j_{k_r+1}j_{k_r}\right)\ldots\left(j_{k_2+l_2}\ldots j_{k_2+1}j_{k_2}\right)\left(j_{k_1+l_1}\ldots j_{k_1+1}j_{k_1}j\right),\end{aligned}$$

with conditions, $j_{k_2} < j_{k_3} < \cdots < j_{k_r}$ and $j_{k_t} < j_{k_t+s}$ for $t = \overline{2, r}$ and $s = \overline{1, l_t}$.