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A numerical method based on fully discrete direct discontinuous Galerkin method for the time fractional diffusion equation

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ABSTRACT

In this paper, an implicit fully discrete direct discontinuous Galerkin (DDG) finite element method is considered for solving the time fractional diffusion equation. The scheme is based on the Gorenflo–Mainardi–Moretti–Paradisi (GMMP) scheme in time and direct discontinuous Galerkin method in space. Unlike the traditional local discontinuous Galerkin method, the DDG method is based on the direct weak formulation for solutions of parabolic equations in each computational cell, letting cells communicate via the numerical flux $\hat{u_x}$ only. We prove that our scheme is stable and the energy norm error estimate is convergent with $O((\Delta x)^k + \Delta t^{\alpha+1} + \Delta t^{\frac{\alpha}{2}}(\Delta x)^k)$ by choosing admissible numerical flux. The DDG method has the advantage of easier formulation and implementation as well as the high order accuracy. Finally numerical experiments are presented to verify our theoretical findings.

1. Introduction

The subject of fractional calculus has gained considerable popularity and importance during the past three decades mainly due to its attractive applications in numerous, seemingly diverse and wide spread fields of science and engineering. Fractional differential equations (FDEs) have been used for mathematical modeling in viscoelastic materials [7], turbulent flow [8], economics [9], etc. Most of the fractional partial differential equations do not have analytic solutions, so solving fractional differential equations numerically becomes significant. However, numerical methods applied must conserve the essential properties of the frictional differential equations. And so far, a variety of such methods have been proposed, for instance, Adomina's decomposition method [17], finite difference method [18,19], discontinuous Galerkin method [4,5], finite element method [20,21], and so on. Applying these numerical methods, various fractional-order partial differential equations have been solved, for example, space–time fractional partial differential equation [10–12], the initial-boundary value time fractional diffusion problem [13–15], the fractional KdV equation [16].

In this paper we consider the following time fractional diffusion equations (TFDEs) in transport process

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t),\ x \in \Omega = (0,L),\ t \in (0,T] \tag{1}$$

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with the following initial and boundary conditions

$$u(x,0) = u_0(x),\ x \in \Omega, \tag{2}$$

$$u(0,t) = 0, u(L,t) = g(t),\ 0 < t \le T, \tag{3}$$

where $\alpha (0 < \alpha \le 1)$ is a parameter describing the order of the fractional time, f, u_0 are given smooth functions. The time fractional diffusion equation is obtained from the standard diffusion equation by replacing the first-order time derivative with a fractional derivative of order $\alpha \in (0, 1)$. From a physical view-point, the generalized diffusion equation is obtained from a fractional Fick law which describes transport processes with long memory.

In order to solve the above time fractional differential equations, Lin and Xu [14] used a finite difference method in time and Legendre spectral method in space. Jiang and Ma [13] proposed a finite element method in space and finite difference method in time. Although many numerical methods for the time fractional partial differential equations have been proposed, it seems that higher convergence order is not easy to obtain.

The discontinuous Galerkin(DG) method is a fascinating method for PDEs because of its flexibility and efficiency in terms of mesh and shape functions, and the higher order of convergence obtained easily. There have been several DG methods suggested in literature to solve the diffusion problem, including the method originally proposed by Bassi and Rebay [23] for compressible Navier–Stokes equations, its generalization called the local discontinuous Galerkin(LDG) method introduced in [24] by Cockburn and Shu and further studied in [25,26]; as well as the method introduced by Baumann–Oden [27,28]. Xu and Zheng [4] presented local discontinuous Galerkin method in space and GMMP scheme in time, and gave the stability and convergence for the fully discrete numerical scheme for this problem.

The key of the local discontinuous Galerkin method for the diffusion equations is to rewrite the equation into a first order system by introducing an auxiliary variable. By solving the system, one obtains the solutions of the diffusion equations. The shortcoming is computational cost larger whether you use explicit and implicit step to solve the fully discrete system by the LDG method. Other DG method is called the direct discontinuous Galerkin (DDG) method introduced by Liu and Yan [2,3] which is based on weak formulation for the solution of the parabolic equation in each computational cell and cells communicate via the numerical flux $\hat{u_x}$ only. Here, it does not introduce any new variables.

The main purpose of this paper is to solve and analyze the time fractional diffusion equations by using a fully discrete direct discontinuous Galerkin method. The scheme is based on GMMP scheme in time and direct discontinuous Galerkin method in space. We firstly prove stability for the scheme by using admissible numerical fluxes. Then we prove the kth order accuracy in an energy norm by using kth degree polynomials. At last, numerical experiments based on the kth degree polynomials with $k = 1, 2, 3$, are presented to demonstrate the optimal $(k + 1)$th order error estimate.

The structure of this paper is as follows. Section 2 begins with introducing some definitions and discrete methods for the fractional order derivative. The DDG scheme for the TFDEs(1), the scheme stability and the error estimation are given in Section 3. The numerical experiments are presented to illustrate the accuracy of the DDG method in Section 4. The paper ends up with concluding remarks included in Section 5.

2. Fractional derivatives and GMMP scheme

In this section, we firstly introduce the definitions of fractional derivatives in three different forms, and then present the GMMP scheme applied in this paper for fractional derivatives.

For simplicity, we consider the interval $[0, t]$ instead of $[a, t]$. Assume that the function $f(x, t)$ satisfies some smoothness conditions in every finite interval $[0, t]$ with $t \le T$. The Riemann–Liuouville fractional derivative of order α reads

$${}_0^{RL}D_t^{\alpha} f(x,t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \dfrac{d^n}{dt^n} \displaystyle\int_0^t \frac{f(x,\tau)}{(t-\tau)^{\alpha+1-n}} d\tau,\ n-1 < \alpha < n, \\ \dfrac{d^n f(t)}{dt^n}, \qquad\qquad \alpha = n, \end{cases} \tag{4}$$

where α satisfies that $n - 1 < \alpha \le n, n \in N$.

The Riemann–Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential proposed by Caputo in his work on the theory of viscoelasticity [1,29]

$${}_0^{C}D_t^{\alpha} f(x,t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t \frac{f^{(n)}(x,\tau)}{(t-\tau)^{\alpha+1-n}} d\tau,\ n-1 < \alpha < n, \\ \dfrac{d^n f(t)}{dt^n}, \qquad\qquad \alpha = n. \end{cases} \tag{5}$$

Rieamnn–Liouville fractional derivative is not equal to Caputo's derivative. Their relation is described by

$${}_0^{C}D_t^{\alpha} f(x,t) = {}_0^{RL}D_t^{\alpha} f(x,t) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha} f^{(k)}(x,0)}{\Gamma(k+1-\alpha)}. \tag{6}$$