Applied Mathematics and Computation 265 (2015) 40–50

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Existence and localization of solutions for operatorial systems defined on Cartesian product of Fréchet spaces using a new vector version of Krasnoselskii's cone compression–expansion theorem

Sz. András[1,*], J.J. Kolumbán[2]

Babeş – Bolyai University, Faculty of Mathematics and Computer Science, Str. M. Kogălniceanu No. 1, Cluj-Napoca 400084, Romania

ARTICLE INFO

Keywords:
Nonlinear system
Differential system
Fréchet space
Fixed points
Positive solution
Componentwise compression–expansion

ABSTRACT

A generalization of Krasnoselskii's compression–expansion fixed point theorem is presented for treating nonlinear systems defined on the Cartesian product of Fréchet spaces. The compression–expansion conditions are given componentwise, and therefore each component can separately behave in its own way. Applications to differential systems of second order on the half line are presented, with existence, localization and multiplicity results.

1. Introduction

The aim of this paper is to study the following type of systems of operator equations

$$\begin{cases} f_1(x_1, x_2) = x_1 \\ f_2(x_1, x_2) = x_2 \end{cases} \tag{1.1}$$

where $(E_1, \{\|\cdot\|_n^{(1)}\}_{n\in\mathbb{N}^*})$, $(E_2, \{\|\cdot\|_n^{(2)}\}_{n\in\mathbb{N}^*})$ are Fréchet spaces and $f = (f_1, f_2) : E_1 \times E_2 \to E_1 \times E_2$. Our approach is based on a new vector version of Krasnoselskii's compression–expansion fixed point theorem for nonlinear operators defined on the Cartesian product of two Fréchet spaces. As an application we present a second order differential system on a non-compact interval, but the abstract results can also be applied to study the existence and localization of solutions for non-local problems [2,11] on non-compact intervals.

The common approach for solving this system would be considering the product space $E_1 \times E_2$, which would also be a Fréchet space with a family of scalar seminorms, formed from the respective seminorms of the two spaces, for instance $\|(x_1, x_2)\|_n = \|x_1\|_n^{(1)} + \|x_2\|_n^{(2)}$. One could then apply existing fixed point results for a single operator equation on a Fréchet space such as those in [4], [5], [6], [7] or [9]. However, in the case of cone compression–expansion results, this leads to problems with the localization of the solution, especially finding lower bounds. This motivated the development of vector versions of such theorems.

* Corresponding author. Tel.: +40 745 561 201.
E-mail addresses: andraszk@yahoo.com, andrasz@math.ubbcluj.ro (Sz. András), jozsi_k91@yahoo.com (J.J. Kolumbán).

[1] The first author was supported by a grant of the Romanian National Authority for Scientific Research, CNCS UEFISCDI, project number PN-II-ID-PCE-2011-3-0094.
[2] The second author was supported by Collegium Talentum.

http://dx.doi.org/10.1016/j.amc.2015.04.124
0096-3003/

Our main fixed point result will incorporate admissible maps in the sense of Frigon [5,6], using the theory of upper semi-continuous multivalued maps. By similar means, we will construct a family of multivalued operators $(\mathbb{F}_n)_{n\geq 1}$, defined on the Cartesian product of appropriately constructed Banach spaces, such that the problem of solving (1.1) will be equivalent to finding for each $n \geq 1$ an element $z_n \in \mathbb{F}_n(z_n)$.

In order to do so, however, first we will give a multivalued generalization of the main result from [10], since we will need to solve the following fixed point problem:

$$\begin{cases} u_1 \in N_1(u_1, u_2) \\ u_2 \in N_2(u_1, u_2) \end{cases}$$

where $N = (N_1, N_2) : K_{rR} \to 2^K$, such that K is a cone in the Cartesian product of two Banach spaces, and K_{rR} is a certain annular region of K. The proof of this theorem will rely on the Kakutani–Glicksberg–Fan fixed point theorem [1].

In [10] Krasnoselskii type theorems (see also [8]) for (1.1) were already discussed in the case of normed linear spaces. One of the motivations for extending those results to Fréchet spaces is the application to differential and integral systems on noncompact intervals. For example, the space $C[0, \infty)$ is not a normed space, but it can be a Fréchet space with a suitable family of seminorms. An example of application to second order differential system will be given in Section 4.

2.2. Preliminaries

In this section we introduce some basic definitions, notations and theorems that shall be used in the remainder of the paper.

Definition 2.1. Let X, Y be topological spaces and $T : D \subseteq X \to Y$.

a) The operator T is said to be *compact* if it maps any bounded subset of D into a relatively compact subset of Y.
b) The operator T is said to be *completely continuous* if it is compact and continuous.

Definition 2.2. Let X, Y be topological spaces and $F : X \to 2^Y$ be a set valued map (that is, for each $x \in X$, $F(x) \in 2^Y$). We say that F is *upper semicontinuous* at $x_0 \in X$ if, for each open set $G \subseteq Y$ with $F(x_0) \subseteq G$, there exists an open neighborhood $N(x_0) \subseteq X$ of x_0 with $F(x) \subseteq G$ for all $x \in N(x_0)$. F is upper semicontinuous on X if it is upper semicontinuous at every point of X.

Definition 2.3. Let X, Y be topological spaces and $F : D \subseteq X \to 2^Y$ be a set valued map.

a) The operator F is said to be *compact* if for any bounded $A \subseteq D$, $F(A) = \bigcup_{a \in A} F(a)$ is a relatively compact subset of Y.
b) The operator F is said to be *completely continuous* if it is compact and continuous.

Lemma 2.4. *Let X, Y be Banach spaces, $K \subset X$ and $F : \overline{K} \to 2^Y$ a set valued map with convex, compact values and closed graph such that F is compact on K. Then it follows that F is compact on $\overline{K}$.*

Proof. Let (x_n) be a bounded sequence in $\overline{K}$, and $y_n \in F(x_n)$, for each $n \in \mathbb{N}^*$. Since F has a closed graph, it follows that, for every $\varepsilon > 0$ and every $(x, y) \in Gr(F)$, there exists $(u, v) \in Gr(F)$, with $u \in K$, such that $||(x, y) - (u, v)|| < \varepsilon$ (for the norm on the product space $X \times Y$ we can simply consider the sum of the two norms on the respective spaces).

Now, if we write this for each $n \in \mathbb{N}^*$ with $\varepsilon = \frac{1}{n}$ for the points (x_n, y_n), it follows that there exists $(u_n, v_n) \in Gr(F)$, with $u_n \in K$, such that $\|(x_n, y_n) - (u_n, v_n)\| < \frac{1}{n}$.

However, this implies that $\|x_n - u_n\| < \frac{1}{n}$, and since (x_n) is bounded, it follows that so is (u_n). Therefore, (u_n) is a bounded sequence in K, $v_n \in F(u_n), \forall n \geq 1$, and F is compact on K, so there exists a convergence sub-sequence (v_{n_k}) of (v_n).

On the other hand, $\|y_n - v_n\| < \frac{1}{n}$, which implies that (y_{n_k}) is a Cauchy sequence. Since Y is a Banach space, it follows that (y_{n_k}) converges, so we may conclude that F is compact on $\overline{K}$. □

Theorem 2.5 (Kakutani–Glicksberg–Fan)**.** *Let K be a nonempty, convex, compact subset of a Hausdorff locally convex linear topological space X and let $F : K \to 2^K$ be upper semicontinuous with nonempty, convex, compact values. Then F has a fixed point, that is, there exists a $y \in K$ with $y \in F(y)$.*

Proof. See [1], pp. 136–138. □

Definition 2.6. Let X be a topological vector space.

a) $K \subseteq X$ is said to be a *cone* if and only if $\lambda K = K$ for every $\lambda \geq 0$.
b) $K \subseteq X$ is said to be a *convex cone* if and only if $\alpha x + \beta y \in K$ for any $x, y \in K$, $\alpha, \beta \geq 0$.
c) $K \subseteq X$ is said to be a *salient cone* if and only if $K \cap (-K) \subseteq \{0\}$.
d) $K \subseteq X$ is said to be a *proper cone* if and only if it is convex and salient.
e) Given a proper cone $K \subseteq X$, we define a strict partial ordering "$\prec$" with respect to K by $x \prec y \Leftrightarrow y - x \in K, x \neq y$.

If, in addition, X is a normed space (let the norm on X be denoted by $|\cdot|$), we consider the following definition as well.

f) The norm on X is said to be *strictly monotonic* if for every $x, y \in K$ $x \prec y$ implies $|x| < |y|$.

For further details on cones and orderings, see [3], from page 218 onwards.