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Superconvergence and *a posteriori* error estimates of the DG method for scalar hyperbolic problems on Cartesian grids

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ABSTRACT

In this paper, we analyze the discontinuous Galerkin (DG) finite element method for the steady two-dimensional transport-reaction equation on Cartesian grids. We prove the L^2 stability and optimal L^2 error estimates for the DG scheme. We identify a special numerical flux for which the L^2-norm of the solution is of order $p+1$, when tensor product polynomials of degree at most p are used. We further prove superconvergence towards a particular projection of the directional derivative. The order of superconvergence is proved to be $p+1/2$. We also provide a very simple derivative recovery formula which is $\mathcal{O}(h^{p+1})$ superconvergent approximation to the directional derivative. Moreover, we establish an $\mathcal{O}(h^{2p+1})$ global superconvergence for the solution flux at the outflow boundary of the domain. These results are used to construct asymptotically exact *a posteriori* error estimates for the directional derivative approximation by solving a local problem on each element. Finally, we prove that the proposed *a posteriori* DG error estimates converge to the true errors in the L^2-norm at $\mathcal{O}(h^{p+1})$ rate and that the global effectivity index converges to unity at $\mathcal{O}(h)$ rate. Our results are valid without the flow condition restrictions. We perform numerical experiments to demonstrate that theoretical rates proved in this paper are optimal.

1. Introduction

In this paper, we present an analysis of the original DG finite element method for the following classical steady linear convection-reaction (or neutron transport) equation

$$\mathbf{a}\cdot\nabla u + cu = f(x,y), \quad (x,y)\in\Omega = [a_1,b_1]\times[a_2,b_2], \tag{1.1a}$$

subject to the Dirichlet boundary conditions on the inflow boundary $\partial\Omega^-$

$$u = \begin{cases} g_1(y), & x = a_1, \quad y\in[a_2,b_2],\\ g_2(x), & x\in[a_1,b_1], \quad y = a_2. \end{cases} \tag{1.1b}$$

Here u is a scalar unknown function, $\mathbf{a}=[\alpha,\beta]^t$ is a given nonzero constant velocity vector, c is an absorption coefficient, and the functions f and g are smooth. Without loss of generality, we assume that $\mathbf{a}$ is unit vector. Let $\partial\Omega=\partial\Omega^+\cup\partial\Omega^-$ denote the boundary of Ω and $\mathbf{n}$ be the outward unit normal vector to $\partial\Omega$. The inflow boundary is $\partial\Omega^-=\{(x,y)\in\partial\Omega|\mathbf{a}\cdot\mathbf{n}<0\}$ and $\partial\Omega^+=\{(x,y)\in\partial\Omega|\mathbf{a}\cdot\mathbf{n}>0\}$. For clarity of presentation, we consider two-dimensional problems, but what follows can be generalized

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for problems in $\mathbb{R}^d$, $d \geq 2$. In our analysis, we assume that c is a positive number. Note that the condition on c can always be fulfilled by a transformation $u = e^{\lambda(\alpha x + \beta y)} v$ for a suitably chosen λ. In fact a straightforward calculation shows that v solves the equation $\mathbf{a} \cdot \nabla v + (c + \lambda) v = \hat{f}$, where $\hat{f} = e^{-\lambda(\alpha x + \beta y)} f$.

The DG method considered here is a class of finite element methods using completely discontinuous piecewise polynomials for the numerical solution and the test functions. It combines many attractive features of the classical finite element and finite volume methods. The DG method was first proposed by Reed and Hill in 1973 [32] for approximating the scalar neutron equation. Since then the method has been analyzed and extended to a wide range of applications. Consult [21,22,24] and the references cited therein for a detailed discussion of the history of DG method and a list of important citations on the DG method and its applications.

The hyperbolic problem (1.1) has been studied by several authors. The first error analysis of the original DG method applied to the steady transport-reaction equation in two space dimensions was carried out in 1974 by LeSaint and Raviart [28]. They established an L^2 convergence rate of order $\mathcal{O}(h^p)$ for approximations by polynomials of degree $\leq p$ and for an arbitrary triangulation, where h is the mesh size. These error estimates are not optimal in the exponent of the parameter h. In fact, numerical examples demonstrated that the error bounds could not be improved in general. However, they proved that the rate of convergence of the DG method is $\mathcal{O}(h^{p+1})$ when quadrilateral elements and tensor product polynomials of degree at most p are used. Later in 1986, Johnson and Pitkaranta [26] proved that the DG solution converges at a rate of convergence of $\mathcal{O}(h^{p+1/2})$ for general triangular meshes and polynomials of degree p. In 1991, Peterson [30] proved that the rate of convergence of $\mathcal{O}(h^{p+1/2})$ is sharp for quasi-uniform triangulation. The mechanisms that induce the loss of $h^{1/2}$ in the order of convergence of the L^2-norm of the error are not very well known yet. In 1988, Richter [33] successfully demonstrated the optimal rate of convergence $\mathcal{O}(h^{p+1})$ for semi-uniform triangle meshes and polynomials of degree p with the assumption that all element edges are bounded away from the characteristic direction of the hyperbolic equation.

Recently, Cockburn et al. [18] showed that the approximation given by the DG method for the transport-reaction equation in d space dimensions is optimal provided the meshes are suitably chosen. They proved that the L^2-norm of the error is of order $p + 1$ when the method uses polynomials of degree p. However, their improved estimates hold for triangulations made of elements satisfying simple flow conditions with respect to the convection direction $\mathbf{a}$. They further find a new element-by-element postprocessing of the derivative in the direction of the flow which superconverges with order $p + 1$. More recently, Zhang et al. [36] analyzed the DG method for hyperbolic problems in d-dimensional space. They proved that the L^2-error estimate is of order $p + 1$ provided the triangulation is made of rectangular elements satisfying some flow conditions. They further established a strong superconvergence at the outflow end of every element where solutions converge at an $\mathcal{O}(h^{2p+1})$ rate. Moreover, they established a derivative recovery formula for the approximation of the convection directional derivative which is superconvergent with order $p + 1$. We also mention the work of Cockburn et al. [19]. They assumed that the triangulation satisfies some flow conditions and proved optimal convergence rates for the approximation on special meshes for variable transport velocity. Finally, they also showed that, by means of an element-by-element postprocessing, a new approximate flux can be obtained which superconverges with order $p + 1$. However, their results hold only for a special class of triangulations.

A posteriori error estimates play an essential role in assessing the reliability of numerical solutions and in developing efficient adaptive algorithms. Several *a posteriori* DG error estimates are known for hyperbolic and convection–diffusion problems [1–15,27,35], to mention a few. Recovery-based error estimators have been studied in [29] and the references therein. A much researched recovery-based error estimator was proposed by Zienkiewicz and Zhu [37], who suggested to post-process the discontinuous gradient in terms of some interpolation functions. The underlying idea is to post-process the gradient and to find an estimate for the true error by comparing the post-processed gradient and the nonpost-processed gradient of the approximation. *A posteriori* error estimators of the recovery-type possess a number of attractive features for the engineering community, because they are easy to implement, computationally simple, asymptotically exact, and produce quite accurate estimates on fine meshes. From a practical point of view, recovery-based error estimators are efficient compared to other implicit residual-based *a posteriori* error estimates. Several recovery-type *a posteriori* error estimators are known for elliptic problems. However, to the author's knowledge, no recovery-type *a posteriori* error estimator for DG methods applied to two-dimensional hyperbolic problems is available in the literature.

In this paper, we analyze the approximation property of the original DG method for solving the steady two-dimensional hyperbolic conservation laws on Cartesian grids. We prove the L^2 stability and optimal L^2 error estimates for the DG scheme. In particular, we identify a special numerical flux for which the L^2-norm of the solution is of order $p + 1$, when tensor product polynomials of degree at most p are used. Our results are valid without the flow condition restrictions. Then, we show that there is a strong superconvergence property at the outflow edges of the whole domain where the discretization error converges as $\mathcal{O}(h^{2p+1})$. We further develop a simple derivative recovery formula which gives a superconvergent approximation to the directional derivative. Our new element-by-element postprocessing of the derivative in the direction of the flow is shown to converge under mesh refinement with order $p + 1$. Based on the superconvergence results, we construct asymptotically exact *a posteriori* error estimates for the directional derivative approximation by solving a local problem on each element and prove that the *a posteriori* DG error estimates converge to the true errors in the L^2-norm under mesh refinement. The order of convergence is proved to be $p + 1$. Finally, we prove that the global effectivity index in the L^2-norm converges to unity at $\mathcal{O}(h)$ rate. Our proofs are valid for arbitrary regular Cartesian meshes using tensor product polynomials of degree at most p. To the best knowledge of the author, our results are novel in the current available literature. The generalization to nonlinear equations involves several technical difficulties and will be investigated in the future.