Applied Mathematics and Computation 265 (2015) 245–265

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A modified integral equation method of the nonlinear elliptic equation with globally and locally Lipschitz source

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ARTICLE INFO

MSC:
35K05
35K99
47J06
47H10

Keywords:
Cauchy problem
Nonlinear elliptic equation
Ill-posed problem
Error estimates

ABSTRACT

The paper is devoted to investigating a Cauchy problem for nonlinear elliptic PDEs in the abstract Hilbert space. The problem is hardly solved by computation since it is severely ill-posed in the sense of Hadamard. We shall use a modified integral equation method to regularize the nonlinear problem with globally and locally Lipschitz source terms. Convergence estimates are established under priori assumptions on exact solution. A numerical test is provided to illustrate that the proposed method is feasible and effective. These results extend some earlier works on a Cauchy problem for elliptic equations

1. Introduction

Let H be a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$, and let A: $D(A) \subset H \to H$ be a positive-definite, self-adjoint operator with a compact inverse on H. Let T be a positive number, we consider the general problem of finding a function u: $[0, T] \to H$ from the system

$$\begin{cases} u_{tt} = Au + f(t, u(t)), t \in (0, T), \\ u(0) = \varphi, \\ u_t(0) = g, \end{cases} \tag{1.1}$$

where the datum g, φ are given in H and the source function f: $[0, T] \times H \to H$ is defined later.

The problem (1.1) is a generalization of several well-known equations. For a simple example, taking $A = -\frac{\partial}{\partial x^2}$ and $f(x, t, u) = -k^2 u$ gives us the Helmholtz equation arising in many engineering applications related to propagating waves in different environments, such as acoustic, hydrodynamic and electromagnetic waves (see, e.g. [2,6]). Furthermore, if we consider the linear case, i.e. the source term does not rely on u, and A the second-order differential operator defined in $H_0^1(\Omega)$

$$Au = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + a(x)u,$$

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http://dx.doi.org/10.1016/j.amc.2015.03.115
0096-3003/

with $a_{ij}, a \in L^\infty(\Omega)$ satisfying

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \le \nu \sum_{i=1}^{n} |\xi_i|^2,$$

where $\nu > 0$ is given, $a(x) \ge 0$, then we obtain the Poisson equation that obviously has been studied in a very long time in both pure and applied aspects, e.g. [12].

Consequently, the inverse problems governed by elliptic PDEs play important roles in many physical and engineering problems. However, there always exists a big question concerning the fact that how to deal with the instability. In fact, such a problem like (1.1) is not well-posed in the sense that a small perturbation in the given Cauchy data (φ, g) may effect a very large error on the solution. Therefore, it is very difficult to solve it using classic numerical methods and proposing regularization methods to overcome the difficulty accordingly occupied a large position in various kinds of studies until now.

In the study of regularization methods, one may find many studies on the homogeneous problem, i.e. $f = 0$ in Eq. (1.1). For instance, Elden and Berntsson [7] used the logarithmic convexity method to obtain a stability result of Hölder type. Alessandrini et al. [1] provided essentially optimal stability results, in wide generality and under substantially minimal assumptions. Reginska and Tautenhahn [16] presented some stability estimates and regularization method for a Cauchy problem for Helmholtz equation. The homogeneous problems were also investigated by some earlier papers, such as [3,4,9,12,15,17,20]. Very recently, the inhomogeneous version of elliptic equations has been considered in [19].

To the authors' knowledge, the results on regularization theory for the Cauchy problem of an elliptic equation with nonlinear source like the problem (1.1) are very rare and until now we do not find any results associated with a locally Lipschitz source term. Additionally, the nonlinear case is, of course, more general and quite close to practical applications than the homogeneous linear case. At once, one may deduce the elliptic-since Gordon equation by taking $f(x, t, u) = \sin u$. In view of physical phenomena models, the motivation for the study of elliptic-sine Gordon equation comes from applications in several areas of mathematical physics including the theory of Josephson effects, superconductors and spin waves in ferromagnets, see e.g. [10,14]. Furthermore, the Lane–Emden equation $\Delta u = mu^p$ (m is constant) plays a vital role in describing the structure of the polytropic stars where p is called the polytropic index. Many abstract studies about this equation are the platform for the system Lane–Emden–Fowler arising in molecular biology, that received considerable mathematical attention, such as Pohozaev-type identities, moving plane method. For more details, the Lane–Emden equation can be referred to the book by Chandrasekhar [5] and Emden [8].

Motivated by all above reasons, in the present paper we propose a new modified method based on a nonlinear integral equation to regularize problem (1.1) with both globally and locally Lipschitz sources. As we know, for a nonlinear Cauchy problem, its solution (exact solution) can be represented in a nonlinear integral equation (called mild solution) which contains some unstable terms. The leading idea of this method is to find a suitable integral equation approximating the exact solution. The work is thus to replace those terms by regularization terms and show that the solution of our regularized problem converges to the solution of the Cauchy problem as the regularization parameter tends to zero. In the homogeneous case, we have many choices of stability term for regularization. However, for the nonlinear problem, the solution u is complicated and defined by an integral equation such that the right hand side depends on u. This leads to studying nonlinear problem is very difficult, so in this paper we develop some new appropriate techniques. For more details, we refer reader to next sections and the papers [11,19].

The paper is organized as follows. In Section 2, the regularization method–integral equation method is introduced. In Section 3, a stability estimate is proved under a priori conditions of the exact solution. In Section 4, a generalized case of the nonlinear problem with a special type of local Lipschitz function is remarkable. Then, a numerical example is shown in Section 5.

2. Mathematical problem and mild solution

As we introduced, $A: D(A) \subset H \to H$ is a positive-definite, self-adjoint operator with a compact inverse on H. Therefore, A admits an orthonormal eigenbasis $\{\phi_p\}_{p \ge 1}$ in H, associated with the eigenvalues such that

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \ldots \lim_{p\to\infty} \lambda_p = \infty,$$

and consequently we have $\langle Au(t), \phi_p\rangle = \lambda_p \langle u(t), \phi_p\rangle$ for all p.

In practice, the data $(\varphi, g) \in H \times H$ is obtained by measuring at discrete nodes. Hence, instead of (φ, g), we shall get an inexact data $(\varphi^\varepsilon, g^\varepsilon) \in H \times H$ satisfying

$$\|\varphi^\epsilon - \varphi\|_H \le \epsilon, \ \|g^\epsilon - g\|_H \le \epsilon, \tag{2.2}$$

where the constant $\varepsilon > 0$ represents a bound on the measurement error.

For the sake of simplicity, we divide the problem (1.1) into three cases: homogeneous linear problem, inhomogeneous linear problem and nonlinear problem.

2.1. Homogeneous linear problem

Let $g, \varphi \in H$, we consider the problem of finding a function $u: [0, T] \to H$ satisfying

$$u_{tt} = Au, \tag{2.3}$$