Applied Mathematics and Computation 265 (2015) 292–303

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

The simplest conforming anisotropic rectangular and cubic mixed finite elements for elasticity ☆

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ARTICLE INFO

MSC:
65N15
65N30

Keywords:
Elasticity
Mixed method
Conforming finite element
Rectangular
Cubic
Anisotropic

ABSTRACT

In this paper, we construct two simplest conforming rectangular elements for the linear elasticity problem under the Hellinger–Reissner variational principle. One is a rectangular element in 2D with only 8 degrees of freedom for the stress and 2 degrees of freedom for the displacement. Another one is a cubic element in 3D with only $18 + 3$ degrees of freedom. We prove that the two elements are stable and anisotropic convergent. Numerical test is presented to illustrate the element is stable and effective.

1. Introduction

In this paper we consider the linear elasticity problem in two and three dimensions based on Hellinger–Reissner variational principle, in which both the stress field and the displacement field are treated as primary unknowns. The design of effective finite elements for this problem proved to be very difficult. The crux lies in the requirements of symmetry imposed on the stress and two stable conditions. Some methods have been presented to this problem, such as in [1–3], the composite element is used in which the displacement space consists of piecewise polynomials in one triangulation domain, while the stress space consists of piecewise polynomials with respect to a more refined element. And in [4–7], the Lagrange functionals are used to modify the Hellinger–Reissner functional in which the symmetry of the stress tensor is enforced only weakly or abandoned. Not until the year 2002, was the first family of stable triangular mixed finite elements [8] proposed for Hellinger–Reissner formulation, using polynomial shape functions with respect to a single arbitrary triangular mesh for both the stress and the displacement spaces. From then on some stable mixed finite elements for the linear elasticity problem are presented. For example, the conforming and nonconforming simplicial elements of two and three dimensions can be found in [9–13]. The nonconforming rectangular elements of two and three dimensions can see [14–18].

In this paper we focus on the conforming rectangular elements of two and three dimensions with strong symmetry for the stress field.

In this direction, for two dimension case, a conforming rectangular element with 45 degrees of freedom for the stress and 12 degrees of freedom for the displacement in [19] is presented, meanwhile a simplified form of this element with 36 + 3 degrees of freedom is given. A conforming rectangular element presented in [20] only has $17 + 4$ degrees of freedom, meanwhile instead

☆ This work was supported by: NSFC (11371331), Science and Technology Research Program of Education Department of Henan Province (N.13A110005).
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http://dx.doi.org/10.1016/j.amc.2015.04.117
0096-3003/

of involving the values of all components of the stress tensor at vertices, only the values of the shear stress at vertices are included. In [21] the degrees of freedom are further reduced, a conforming rectangular element with $10+4$ degrees of freedom is constructed.

For three dimension case, in [22] a conforming cubic element with 72 degrees of freedom for the stress and 6 degrees of freedom for the displacement is presented. And a conforming cubic element reported in [21] only has 21 + 6 degrees of freedom.

By the way, all above elements which have been reported are assumed that the mesh is shape regular [23,24].

In this paper we present a rectangular conforming element only with 8 + 2 degrees of freedom and a cubic conforming element only with $18+3$ degrees of freedom. They are the simplest conforming rectangular elements in 2D and 3D. Meanwhile they are anisotropic convergent, the mesh shape regular condition is not needed.

This paper is organized as follows. In Section 2, we collect the notations used in the paper. In Section 3, we first construct the rectangular conforming element R8-2 and the cubic conforming element C18-3. Then we prove the well-posedness and stability of the finite elements R8-2 and C18-3. In Section 4, we show the optimal order convergence of these two elements without the mesh shape regular condition. And in Section 5, we give a numerical example which illustrates the proper convergence of this element. Section 6 is a conclusion of this paper.

2. Notations and preliminaries

Let $\boldsymbol{v}=(v_1,\ldots,v_d)^T$ is a vector-valued field of d variables, $d=2$ or 3, then its gradient denoted by grad $\boldsymbol{v}$, is the matrix-valued function obtained by applying the ordinary gradient operator to each component of $\boldsymbol{v}$. Similarly, when $\boldsymbol{\tau}=(\tau_{ij})_{d\times d}$ is a tensor, its divergence denoted by $\operatorname{div}\boldsymbol{\tau}$, is the vector field obtained by applying the divergence operator to each row of $\boldsymbol{\tau}$. That is

$$\operatorname{grad} v=\begin{pmatrix}\dfrac{\partial v_1}{\partial x_1} & \dfrac{\partial v_1}{\partial x_2}\\ \dfrac{\partial v_2}{\partial x_1} & \dfrac{\partial v_2}{\partial x_2}\end{pmatrix},\qquad \operatorname{div}\boldsymbol{\tau}=\begin{pmatrix}\dfrac{\partial \tau_{11}}{\partial x_1}+\dfrac{\partial \tau_{12}}{\partial x_2}\\ \dfrac{\partial \tau_{21}}{\partial x_1}+\dfrac{\partial \tau_{22}}{\partial x_2}\end{pmatrix},\qquad d=2.$$

$$\operatorname{grad} v=\begin{pmatrix}\dfrac{\partial v_1}{\partial x_1} & \dfrac{\partial v_1}{\partial x_2} & \dfrac{\partial v_1}{\partial x_3}\\ \dfrac{\partial v_2}{\partial x_1} & \dfrac{\partial v_2}{\partial x_2} & \dfrac{\partial v_2}{\partial x_3}\\ \dfrac{\partial v_3}{\partial x_1} & \dfrac{\partial v_3}{\partial x_2} & \dfrac{\partial v_3}{\partial x_3}\end{pmatrix},\qquad \operatorname{div}\boldsymbol{\tau}=\begin{pmatrix}\displaystyle\sum_{j=1}^{3}\dfrac{\partial \tau_{1j}}{\partial x_j}\\ \displaystyle\sum_{j=1}^{3}\dfrac{\partial \tau_{2j}}{\partial x_j}\\ \displaystyle\sum_{j=1}^{3}\dfrac{\partial \tau_{3j}}{\partial x_j}\end{pmatrix},\qquad d=3.$$

In this paper, $\|\cdot\|_{0,T}$ is denoted the norm of space $L^2(T)$, and $H^k(T,X)$ denotes the space of functions on $T\subset\mathbb{R}^d$, with values in the finite-dimensional space X and all derivatives of order at most k in $L^2(T)$. If X ranges over $\mathbb{R}$, we will simply use $H^k(T)$ instead. The norm and semi-norm in $H^k(T,X)$ is denoted by $\|\cdot\|_{k,T}$ and $|\cdot|_{k,T}$. That is

$$\|u\|_{k,T}=\left(\sum_{0\le|\alpha|\le k}\|D^\alpha u\|_{0,T}^2\right)^{\frac12},\quad |u|_{k,T}=\left(\sum_{|\alpha|=k}\|D^\alpha u\|_{0,T}^2\right)^{\frac12}.$$

where the $D^\alpha u$ is the weak partial derivative of function u. The space $P_{k_1,k_2}(T,\mathbb{R})$ (k_1, k_2 are integers), always written as P_{k_1,k_2}, consists of polynomials on T of degree at most k_1 in x_1 and k_2 in x_2. We set $P_{k_1,k_2}=:\{0\}$, if either k_1 or k_2 is negative. The same rule also holds for $P_k(T,\mathbb{R})$, the space of polynomials in two variables x_1 and x_2 of total degree at most k. These symbols are naturally generalized to 3-dimension case. Let $\mathbb{S}$ denote the space of symmetric tensor. The space $L^2(T,\mathbb{R}^d)$ is the set of vector-valued functions which are square integrable. The space $H(\operatorname{div},T,\mathbb{S})$ consists of square-integrable symmetric matrix fields with square-integrable divergence. The norm $\|\cdot\|_{H(\operatorname{div},T)}$ is defined as

$$\|\boldsymbol{\tau}\|_{H(\operatorname{div},T)}^2=\|\boldsymbol{\tau}\|_{0,T}^2+\|\operatorname{div}\boldsymbol{\tau}\|_{0,T}^2.$$

Supposed that an elastic body, occupying a simple connected polygonal domain Ω of $\mathbb{R}^d$ and clamped on $\partial\Omega$, is subject to the given body force f, then the continuous problem is: find $(\boldsymbol{\sigma},\boldsymbol{u})\in H(\operatorname{div},\Omega,\mathbb{S})\times L^2(\Omega,\mathbb{R}^d)$ such that

$$\begin{cases}a(\boldsymbol{\sigma},\boldsymbol{\tau})+b(\boldsymbol{\tau},\boldsymbol{u})=0, & \forall\boldsymbol{\tau}\in H(\operatorname{div},\Omega,\mathbb{S}),\\ b(\boldsymbol{\sigma},\boldsymbol{v})=(f,\boldsymbol{v}), & \forall\boldsymbol{v}\in L^2(\Omega,\mathbb{R}^d),\end{cases}\tag{2.1}$$

where $a(\boldsymbol{\sigma},\boldsymbol{\tau})=\int_\Omega A\boldsymbol{\sigma}:\boldsymbol{\tau}\,dx$, $b(\boldsymbol{\tau},\boldsymbol{v})=\int_\Omega \operatorname{div}\boldsymbol{\tau}\cdot\boldsymbol{v}\,dx$, $(f,\boldsymbol{v})=\int_\Omega f\cdot\boldsymbol{v}\,dx$. Throughout the paper, the compliance tensor $A=A(x):\mathbb{S}\to\mathbb{S}$, characterizing the properties of the material, is bounded and symmetric positive definite uniformly for $x\in\Omega$.

3. The 2-D and 3-D conforming rectangular finite element spaces

In this section, we use mixed finite element method to solve problem (2.1). Two elements are given, one is in two dimensional space and another one is in three dimensional space.