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Rationalized Haar wavelet bases to approximate solution of nonlinear Fredholm integral equations with error analysis

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ABSTRACT

In this article we approximate the solutions of the nonlinear Fredholm integral equations of the second kind, by the method based on using the properties of RH wavelets and matrix operator. Also, the Banach fixed point theorem guarantees the convergence of the method. Also we get an upper bound for the error. Furthermore, the order of convergence is analyzed. The algorithm to compute the solutions and some numerical examples are also illustrated. The numerical results obtained by our method have been compared with other methods.

1. Introduction and preliminaries

Analytical solutions of integral equations either do not exist or are hard to find. So, many numerical methods have been developed to find the solutions of nonlinear integral equations. The use of wavelets has had a lot of prominence during the last two decades. Wavelets can be used as analytical tools for signal processing, numerical analysis and mathematical modeling.

The early works concerning wavelets were in the 1980s by Morlet, Grossmann, Meyer, Mallat and others [5,7,15,16]. But in fact, it was the paper of Daubechies in 1988 that caught the attention of the applied mathematics communities in signal processing, and numerical analysis. Many of the early works were discussed in [5] and [6,15]. The goal of the most modern wavelet research is to create a set of basis functions and transform them that leads to an informative and useful description of a function or signal. Various types of wavelets, such as Haar, Legendre, trigonometric, CAS, Chebyshev, and Coifman wavelets, have been used to approximate the solutions of different types of integral equations. For example, Lepik and Tamme in [8] have used Haar wavelets to the nonlinear Fredholm integral equations, although their method involves approximation of certain integrals. some applications of Fredholm integral equations can be found in [17–19].

The orthogonal set of Haar functions is a group of square waves with a magnitude of $+2^{i/2}$, $-2^{i/2}$ and 0, for $i = 0, 1, \dots$ [12]. Lynch and Reis [9] have rationalized Haar transform by omitting the irrational numbers and introducing the integral powers of two. This modification is called the rationalized Haar (RH) transform. RH transform preserves all the properties of the original Haar transform and can be efficiently implemented by using digital pipeline architecture [14]. The corresponding functions are known as RH functions. The RH functions are composed of only three amplitudes $+1$, -1 and 0.

In [13], the Haar wavelet method and the rationalized Haar function method have been used to solve the nonlinear Fredholm Hammerstein integral equations. The aim of this work is to present a numerical method to approximate the solutions of the

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nonlinear Fredholm integral equations of the second kind as follows:

$$\lambda u(t) = f(t) + \int_0^1 K(t, s, u(s))ds, \qquad u \in X := C([0, 1], \mathbb{R}), \tag{1}$$

where $\lambda \in \mathbb{R} - \{0\}$, $f : [0, 1] \to \mathbb{R}$ and $K : [0, 1]^2 \times \mathbb{R} \to \mathbb{R}$ are assumed to be known continuous functions, and the unknown function that has to be determined is $u : [0, 1] \to \mathbb{R}$. The integral operator T: $(X, \|.\|_\infty) \to (X, \|.\|_\infty)$ is also defined as

$$(Tu)(t) = \frac{1}{\lambda} f(t) + \frac{1}{\lambda} \int_0^1 K(t, s, u(s))ds, \qquad t \in [0, 1], u \in X. \tag{2}$$

The Banach fixed point theorem guarantees that under certain assumptions [1], T has a unique fixed point; which means, the Fredholm integral equation has exactly one solution. Moreover, if we assume that K is a Lipschitz function at its third variable with the Lipschitz constant $M > 0$ and $|\lambda| > M$, then the operator T is contractive with contraction number $N = M/|\lambda|$, so T has a unique fixed point φ. Furthermore, $\varphi = \lim_{n\to\infty} T^n(\varphi_0)$, where φ_0 is any continuous function on [0, 1]. In general, calculating φ, and the sequence of functions $\{T^n(\varphi)\}_{n\in\mathbb{N}}$ explicitly, is not possible so define a new sequence of functions, denoted by $\{\varphi_h\}_{h\in\mathbb{N}}$, obtained recursively making use of RH basis. More concretely, we can get φ_{h+1} from φ_h, approximating $T(\varphi_h)$ by means of the sequence of projections of such RH basis.

2. Properties of the rationalized Haar functions

Definition 2.1. RH wavelet is a function defined on the real line $\mathbb{R}$ as follows:

$$\mathrm{RH}(t) = \begin{cases} 1, & 0 < t \le \dfrac{1}{2}, \\ -1, & \dfrac{1}{2} < t < 1, \\ 0, & \text{otherwise}. \end{cases} \tag{3}$$

Definition 2.2. RH functions $h_n(t)$ for $n = 1, 2, \ldots,$ are defined by

$$h_n(t) = \mathrm{RH}(2^j t - k). \tag{4}$$

where $n = 2^j + k$, with $j = 0, 1, \ldots$ and $k = 0, 1, \ldots, 2^j - 1$.

We know that $h_0(t) = 1$ on interval $t \in [0, 1]$, which is called the scaling function. Here, 2^j, for $j = 0, 1, \ldots,$ indicates the level of the wavelet and $k = 0, 1, \ldots, 2^j - 1$ is the translation parameter. Also, the square of every RH function is a block-pulse with magnitude 1 during both positive and negative half-waves of RH function. It can be shown that the sequence $\{h_n\}_{n=0}^\infty$ is a complete orthogonal system in $L^2[0, 1]$ and for $f \in C[0, 1]$, the series

$$\sum_n 2^j \langle f, h_n\rangle h_n,$$

where

$$\langle f, h_n\rangle = \int_0^1 f(t)h_n(t)dt,$$

converges uniformly to f, see e.g. [16]. Note that the orthogonality property is

$$\int_0^1 h_m(t)h_n(t)dt = 2^j\delta_{n,m},$$

where $n = 2^j + k$ and $m = 2^i + k'$ with $i, j = 0, 1, \ldots,$ and $k = 0, 1, \ldots, 2^j - 1$ and $k' = 0, 1, \ldots, 2^i - 1$.

3. Numerical approximation of the solutions

In this paper we have used the successive approximations method for (1), with initial condition $u_0 \in C[0, 1]$, (usually $f(x)$). This iterative process will continue until it reaches to the error, which usually occurs in the 5th or 6th iteration. For every $t, s \in [0, 1]$ and $i \ge 1$ and $m \in \mathbb{N}$, we define the following recursively

$$\psi_{i-1}(t, s) := K(t, s, u_{i-1}(s)). \tag{5}$$

That

$$u_i(t) := \frac{1}{\lambda} f(t) + \frac{1}{\lambda} \int_0^1 Q_m(\psi_{i-1})(t, s)ds, \tag{6}$$