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A new generalized parameterized inexact Uzawa method for solving saddle point problems☆

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ABSTRACT

Recently, Bai, Parlett and Wang presented a class of parameterized inexact Uzawa (PIU) methods for solving saddle point problems (Bai et al., 2005). In this paper, we develop a new generalized PIU method for solving both nonsingular and singular saddle point problems. The necessary and sufficient conditions of the convergence (semi-convergence) for solving nonsingular (singular) saddle point problems are derived. Meanwhile, the characteristic of eigenvalues of the iteration matrix corresponding to the above iteration method is discussed. We further show that the generalized PIU-type method proposed in this paper has a wider convergence (semi-convergence) region than some classical Uzawa methods, such as the inexact Uzawa method, the SOR-like method, the GSOR method and so on. Finally, numerical examples are given to illustrate the feasibility and efficiency of this method.

1. Introduction

Consider the large, sparse system of linear equations

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}, \tag{1.1}$$

where $A \in R^{n \times n}$ is symmetric positive definite, $B \in R^{m \times n}$, and given vectors $p \in R^n$, $q \in R^m$, $(m \leq n)$, B^T denotes the transpose of matrix B.

If the sub-matrix B in (1.1) is full row rank, the system of linear equations (1.1) is called a nonsingular saddle point (NSP) problem, otherwise it is called a singular saddle point (SP) problem (see, e.g., [1,2]). Many practical problems arising from scientific computing and engineering applications may require the solution of the linear systems (1.1), such as computational fluid dynamics [3], optimization and control [4], linear elasticity and mixed finite element method of elliptic partial differential equations [5], or the generalized least squares problems [6], electronic networks, computer graphics and so on.

Generally speaking, the iterative methods, because of storage requirements and preservation sparsity, are more attractive than direct method, although the direct method plays an important role in the form of preconditioners embedded in an iterative framework (see, e.g., [7,8]). Many kinds of efficient iterative methods as well as their numerical properties have been studied in

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the literature, such as Uzawa-type methods [7,9–13], preconditioned Krylov subspace iteration methods [14], HSS iteration methods [15,16], preconditioned conjugate gradient methods [17,18] and others. It is worth mentioning that the parameterized inexact Uzawa (PIU) method put forward by Bai et al. in [7], also termed as the generalized successive overrelaxation (GSOR) method, is feasible and effective for solving the large sparse NSP problem (1.1), including the classical Uzawa method [7] and the SOR-like method [26] as special cases. After that many authors have developed the Uzawa method; see, e.g., [19–21,23,25,26,30,31,35,36] and therein.

Although the sub-matrix B in (1.1) often occurs in the form of full row rank, but not always. Hence, how to effectively solve the SP problem (1.1) is an important but interesting thing. Recently, a preconditioned minimum residual (P-MINRES) and a preconditioned conjugate gradient (PCG) methods were proposed in [39,40], respectively, for solving the SP problem. Bai in [34] firstly proposed a class of Hermitian and skew-Hermitian splitting methods for singular linear systems, and discussed the semi-convergence of this method. After that several authors applied this method to singular saddle point problems, and established the associated semi-convergence conditions. For instance, Zheng et al. [24] have derived the semi-convergence of parameterized Uzawa methods. Zhang et al. [22] have discussed the semi-convergence of the inexact parameterized Uzawa methods, which extends the results presented in [24]. Moreover, Fan and Yang proposed GLHSS method and Uzawa-HSS method for solving non-Hermitian SP problem in [37] and [38], respectively. For more literature on this theme, one can refer to [32,33] and references therein.

Motivated by the afore-mentioned parameterized Uzawa methods, in this paper, we will propose a kind of new parameterized inexact Uzawa-type (abbreviated by PIU-type) methods for solving nonsingular and singular saddle point problems, and then investigate in depth the convergent and semi-convergent conditions of this iteration method, respectively. By choosing different parameter matrices, we will show that the PIU-type method covers a series of existing iterative methods, including the preconditioned Uzawa method, the inexact Uzawa method, parameterized inexact Uzawa method and so on. Particularly, the splitting of the coefficient matrix results in an efficient preconditioner for Krylov subspace method. Numerical experiments show that, compared with the generalized minimal residual (GMRES) method, the corresponding preconditioned GMRES methods based on the PIU-type method is feasible and effective, which may outperform some forgone preconditioners by choosing suitable parameters.

This paper is organized as follows. In Section 2, we will present the PIU-type method for solving saddle point problem (1.1), which is the extensions of several parameterized inexact Uzawa-type methods. In Section 3, the necessary and sufficient conditions of the convergence for solving nonsingular saddle point problem will be obtained. The expressions of eigenvalues of the iteration matrix of the above method will be given in Section 4. Furthermore, the semi-convergence conditions of the PIU-type method for solving singular saddle point problem will be discussed in Section 5. In Section 6, some numerical examples will be afforded to illustrate the feasibility and efficiency of this method. Finally, a conclusion will be drawn to end this paper.

2. The PIU-type method

In this section, we firstly introduce a new generalized parameterized inexact Uzawa-type method to solve the saddle point problem (1.1). It will be shown that this method covers several Uzawa-type methods under suitable situations. For the sake of simplicity, the saddle point problem (1.1) can be rewritten as

$$\begin{bmatrix} A & B^T \\ -B & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} p \\ -q \end{bmatrix}. \tag{2.1}$$

Let

$$\mathscr{A} := \begin{bmatrix} A & B^T \\ -B & 0 \end{bmatrix}, \ \mathcal{Z} := \begin{bmatrix} u \\ v \end{bmatrix}, \ \mathcal{F} := \begin{bmatrix} p \\ -q \end{bmatrix},$$

then (2.1) can be expressed as

$$\mathscr{A}\mathcal{Z} = \mathcal{F}. \tag{2.2}$$

For the coefficient matrix $\mathscr{A}$ of the linear equations (2.1), we make the following splitting:

$$\mathscr{A} = \begin{bmatrix} \frac{1}{\omega}(A+Q_1) & 0 \\ -B+Q_3 & \frac{1}{\tau}Q_2 \end{bmatrix} - \begin{bmatrix} \frac{1}{\omega}(A+Q_1)-A & -B^T \\ Q_3 & \frac{1}{\tau}Q_2 \end{bmatrix} := \mathcal{M}-\mathcal{N}, \tag{2.3}$$

where $Q_1 \in R^{n\times n}$ and $Q_2 \in R^{m\times m}$ are, respectively, prescribed positive semi-definite and positive definite matrices, and $Q_3 \in R^{m\times n}$ is an arbitrary matrix. Then we derive the following so-called parameterized inexact Uzawa-type method:

Method 2.1 (PIU-type method)**.** Let $Q_1 \in R^{n\times n}$ and $Q_2 \in R^{m\times m}$ be prescribed symmetric positive semi-definite and symmetric positive definite matrices, respectively, and $Q_3 \in R^{m\times n}$ be an arbitrary matrix. Given initial vectors $u^{(0)} \in R^n$ and $v^{(0)} \in R^m$, and two relaxation factors $\omega > 0$ and $\tau > 0$. For $k = 0, 1, 2, \ldots$ until the iteration sequence $\{(u^{(k)T}, v^{(k)T})^T\}$ converges to the exact