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Volumetric barrier decomposition algorithms for stochastic quadratic second-order cone programming

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ABSTRACT

Ariyawansa and Zhu (2011) have derived volumetric barrier decomposition algorithms for solving two-stage stochastic semidefinite programs and proved polynomial complexity of certain members of the algorithms. In this paper, we utilize their work to derive volumetric barrier decomposition algorithms for solving two-stage stochastic convex quadratic second-order cone programming, and establish polynomial complexity of certain members of the proposed algorithms.

1. Introduction

The purpose of this paper is to derive a class of volumetric barrier decomposition-based interior point algorithms for two-stage stochastic convex *quadratic second-order cone* programs (SQSOCPs) with recourse in the dual standard form:

$$\begin{array}{ll} \max & \eta(\boldsymbol{x}) := \frac{1}{2}\boldsymbol{x}^{\mathsf{T}}C\boldsymbol{x} + \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} + \mathbb{E}\left[Q(\boldsymbol{x},\omega)\right] \\ \text{s.t.} & A\boldsymbol{x} + \boldsymbol{\xi} = \boldsymbol{b}, \\ & \boldsymbol{\xi} \in \hat{\mathcal{E}}_+, \end{array} \tag{1}$$

where $\boldsymbol{x}$ and $\boldsymbol{\xi}$ are the first-stage decision variables, and $Q(\boldsymbol{x}, \omega)$ is the maximum value of the problem

$$\begin{array}{ll} \max & \frac{1}{2}\boldsymbol{y}(\omega)^{\mathsf{T}}D(\omega)\boldsymbol{y}(\omega) + \boldsymbol{d}(\omega)^{\mathsf{T}}\boldsymbol{y}(\omega) \\ \text{s.t.} & W(\omega)\boldsymbol{y}(\omega) + \boldsymbol{\zeta}(\omega) = \boldsymbol{\hbar}(\omega) - T(\omega)\boldsymbol{x}, \\ & \boldsymbol{\zeta}(\omega) \in \check{\mathcal{E}}_+, \end{array} \tag{2}$$

where $\boldsymbol{y}(\omega)$ and $\boldsymbol{\zeta}(\omega)$ are the second-stage variables, $\mathbb{E}[Q(\boldsymbol{x},\omega)] := \int_\Omega Q(\boldsymbol{x},\omega)P(d\omega)$, the matrices $C \in \mathbb{R}^{n\times n}$ and $A \in \mathbb{R}^{s\times n}$ and the vectors $\boldsymbol{b} \in \mathbb{R}^s$ and $\boldsymbol{c} \in \mathbb{R}^n$ are deterministic data, and the matrices $D(\omega) \in \mathbb{R}^{m\times m}$, $W(\omega) \in \mathbb{R}^{l\times m}$ and $T(\omega) \in \mathbb{R}^{l\times n}$ and the vectors $\boldsymbol{\hbar}(\omega) \in \mathbb{R}^l$ and $\boldsymbol{d}(\omega) \in \mathbb{R}^m$ are random data whose realizations depend on an underlying outcome ω in an event space Ω with a known probability function P. The matrices C and $D(\omega)$ are symmetric positive semidefinite, $\hat{\mathcal{E}}_+$ is a Cartesian product of r_1 second-order cones of dimensions $\hat{d}_1, \hat{d}_2, \ldots, \hat{d}_{r_1}$ with $s = \sum_{i=1}^{r_1} \hat{d}_i$, and $\check{\mathcal{E}}_+$ is a Cartesian product of r_2 second-order cones of dimensions $\check{d}_1, \check{d}_2, \ldots, \check{d}_{r_2}$ with $l = \sum_{j=1}^{r_2} \check{d}_j$.

In stochastic second-order cone programs we handle uncertainty in data defining deterministic second-order cone programs to model many examples of an uncertain data arising in real-word applications. A wide variety of optimization problems can

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be modeled as stochastic second-order cone programs. This includes, but not limited to: Stochastic Euclidean facility location problem, the portfolio optimization problem with loss risk constraints, the optimal covering random ellipsoid problem, and stochastic routing problem in mobile ad hoc networks (see [1,2]). Furthermore, the use of convex quadratic objective together with second-order cone constraints provides a more general problem setting and is hence promising to be applicable to a wider variety of real-world applications.

The work in [3] derives a *logarithmic* barrier decomposition-based interior point algorithm for SQSOCPs by utilizing the work of Alzalg and Ariyawansa [4] for stochastic symmetric programs. In this paper, we derive a *volumetric* barrier decomposition-based interior point algorithm for SQSOCPs by utilizing the work of Ariyawansa and Zhu [5] for stochastic semidefinite programs. Since the linear inequality is a particular case of the second-order cone inequality, our result may be viewed as an extension of the work of Ariyawansa and Zhu [6] for stochastic quadratic linear programs. In this paper, we also take the advantage of the work of Alzalg and Ariyawansa [7] and Schmieta [8] and use the Jordan algebraic characterization of the second-order cone to give explicit expressions for gradient and Hessian computations for the volumetric barrier function. We also establish polynomial complexity of the resulting algorithm.

We organize this paper as follows: In Section 2, we introduce the formulation of the problem, some assumptions, and the volumetric barrier for our problem formulation. In Section 3, we show that the set of volumetric barrier functions for positive values of barrier parameter comprises a self-concordant family. Based on this property, short- and long-step variants of an interior point decomposition algorithm and its convergence theorems are presented in Section 4. Section 5 has some concluding remarks. In Appendix A, we briefly review some notations and background of the algebra of the second-order cone used in the paper. In Appendix B, we prove the convergence theorems stated in Section 4. Before proceeding, the reader is encouraged to read the material presented in Appendix A.

2. Problem formulation and assumptions

In this section, we formulate appropriate volumetric barrier for the SQSOCP problems (1) and (2) (with finite event space Ω) and obtain expressions for the derivatives required in the rest of the paper.

We examine Problems (1) and (2) when the event space Ω is finite. Let

$$\left\{\left(D^{(k)}, T^{(k)}, W^{(k)}, \boldsymbol{\hbar}^{(k)}, \boldsymbol{d}^{(k)}\right) : k = 1, 2, \ldots, K\right\}$$

be the set of the possible values of the random variables $(D(\omega), T(\omega), W(\omega), \hbar(\omega), \boldsymbol{d}(\omega))$ and let

$$p_k := P(D(\omega), T(\omega), W(\omega), \boldsymbol{\hbar}(\omega), \boldsymbol{d}(\omega)) = \left(D^{(k)}, T^{(k)}, W^{(k)}, \boldsymbol{\hbar}^{(k)}, \boldsymbol{d}^{(k)}\right)$$

be the associated probability for $k = 1, 2, \ldots, K$. Then, using the notations introduced in Appendix A, Problems (1) and (2) become

$$\begin{array}{ll} \max & \eta(\boldsymbol{x}) := \frac{1}{2}\boldsymbol{x}^{\mathsf{T}} C \boldsymbol{x} + \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} + \sum_{k=1}^{K} p_k Q^{(k)}(\boldsymbol{x}) \\ \text{s.t.} & A\boldsymbol{x} \preceq \boldsymbol{b}, \end{array} \tag{3}$$

where for $k = 1, 2, \ldots, K,\ Q^{(k)}(\boldsymbol{x})$ is the maximum value of the problem

$$\begin{array}{ll} \max & \frac{1}{2}\boldsymbol{y}(\omega)^{\mathsf{T}} D(\omega)\boldsymbol{y}(\omega) + \boldsymbol{d}(\omega)^{\mathsf{T}}\boldsymbol{y}(\omega) \\ \text{s.t.} & W(\omega)\boldsymbol{y}(\omega) \preceq \boldsymbol{\hbar}(\omega) - T(\omega)\boldsymbol{x}, \end{array} \tag{4}$$

Note that the constraints in ([3,4]) are non-positive second-order cone constraints while the common practice in the deterministic second-order cone programming literature is to use non-negative second-order cone constraints. So for convenience we redefine $\boldsymbol{d}^{(k)}$ as $\boldsymbol{d}^{(k)} := p_k\boldsymbol{d}^{(k)}$ for $k = 1, 2, \ldots, K$, and rewrite Problems (3) and (4) as

$$\begin{array}{ll} \min & \frac{1}{2}\boldsymbol{x}^{\mathsf{T}} C \boldsymbol{x} + \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} + \sum_{k=1}^{K} Q^{(k)}(\boldsymbol{x}) \\ \text{s.t.} & A\boldsymbol{x} - \boldsymbol{b} \succeq \boldsymbol{0}, \end{array} \tag{5}$$

where for $k = 1, 2, \ldots, K,\ Q^{(k)}(\boldsymbol{x})$ is the minimum value of the problem

$$\begin{array}{ll} \min & \frac{1}{2}\boldsymbol{y}^{(k)\mathsf{T}} D^{(k)}\boldsymbol{y}^{(k)} + \boldsymbol{d}^{(k)\mathsf{T}}\boldsymbol{y}^{(k)} \\ \text{s.t.} & W^{(k)}\boldsymbol{y}^{(k)} + T^{(k)}\boldsymbol{x} - \boldsymbol{\hbar}^{(k)} \succeq \boldsymbol{0}. \end{array} \tag{6}$$

In the rest of this paper our attention will be on Problems (5) and (6), and from now on when we use the acronym "SQSOCP" in this paper we mean Problems (5) and (6).