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On developing fourth-order optimal families of methods for multiple roots and their dynamics[☆]

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ABSTRACT

There are few optimal fourth-order methods for solving nonlinear equations when the multiplicity m of the required root is known in advance. Therefore, the first focus of this paper is on developing new fourth-order optimal families of iterative methods by a simple and elegant way. Computational and theoretical properties are fully studied along with a main theorem describing the convergence analysis. Another main focus of this paper is the dynamical analysis of the rational map associated with our proposed class for multiple roots; as far as we know, there are no deep study of this kind on iterative methods for multiple roots. Further, using Mathematica with its high precision compatibility, a variety of concrete numerical experiments and relevant results are extensively treated to confirm the theoretical development.

1. Introduction

With the advancement of computer algebra, finding higher-order multi-point methods, not requiring the computation of second-order derivative for multiple roots become very important and interesting task from the practical point of view. These multi-point methods are of great practical importance since they overcome theoretical limits of one-point methods concerning the order and computational efficiency. Further, these multi-point iterative methods are also capable to generate root approximations of high accuracy.

Let us consider a nonlinear function $f : D \subset \mathbb{R} \to \mathbb{R}$, where D is an open interval such that $r_m \in D$ is a root of equation $f(x) = 0$ with multiplicity m.

In the last years, some optimal iterative methods (in the sense of Kung–Traub conjecture [1]) have appeared. In 2009, Li et al. [2] proposed the following fourth-order optimal two-point method which requires one function and two first-order derivative evaluations per iteration

$$\begin{cases} y_n = x_n - \dfrac{2m}{m+2}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{f(x_n)}{2f'(x_n)}\left[\dfrac{m(m-2)\left(\frac{m}{m+2}\right)^m f'(y_n) - m^2 f'(x_n)}{f'(x_n) - \left(\frac{m}{m+2}\right)^m f'(y_n)}\right]. \end{cases} \tag{1.1}$$

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Sharma and Sharma [3] proposed the following optimal variant of Jarratt's method for obtaining multiple roots

$$\begin{cases} y_n = x_n - \dfrac{2m}{m+2}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{m}{8}\left\{(m^3-4m+8) - (m+2)^2\left(\dfrac{m}{m+2}\right)^m \dfrac{f'(x_n)}{f'(y_n)} \right. \\ \qquad \left. \times\left(2(m-1) - (m+2)\left(\dfrac{m}{m+2}\right)^m \dfrac{f'(x_n)}{f'(y_n)}\right)\right\}\dfrac{f(x_n)}{f'(x_n)}. \end{cases} \tag{1.2}$$

It has fourth-order of convergence and requires one-function and two-derivative evaluation per iteration.

Again in 2010, Li et al. [4] proposed six fourth-order two-point methods with closed formulas for finding multiple zeros of nonlinear functions. Among them, the following one is optimal:

$$\begin{cases} y_n = x_n - \dfrac{2m}{m+2}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - a_3\dfrac{f(x_n)}{f'(y_n)} - \dfrac{f(x_n)}{b_1 f'(x_n) + b_2 f'(y_n)}, \end{cases} \tag{1.3}$$

where

$$\begin{aligned} a_3 &= -\frac{m(m+2)(m+2)^3}{2(m^3-4m+8)}\left(\frac{m}{m+2}\right)^m, \\ b_1 &= -\frac{(m^3-4m+8)^2}{m(m^2+2m-4)^3}, \\ b_2 &= \frac{m^2(m^3-4m+8)}{(m^2+2m-4)^3}\left(\frac{m+2}{m}\right)^m. \end{aligned}$$

Zhou el al. [5] in 2011 constructed a more general iteration scheme for multiple roots, requiring one function and two derivative evaluation per iteration as follows:

$$\begin{cases} y_n = x_n - \dfrac{2m}{m+2}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} Q\left(\dfrac{f'(y_n)}{f'(x_n)}\right), \end{cases} \tag{1.4}$$

where $Q(\cdot) \in C^2(\mathbb{R})$ is a weight function and discussed the conditions on Q to obtain fourth-order optimal methods from it. Zhou et al. have also proved that the above methods namely, (1.2) and (1.3) are special cases of his scheme.

In 2012, Sharifi et al. [6], proposed an optimal family of fourth-order methods as below

$$\begin{cases} y_n = x_n - \dfrac{2m}{m+2}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n + \left(\dfrac{m(m^2+2m-4)}{4}\dfrac{f(x_n)}{f'(x_n)} - \dfrac{m(m+2)^2}{4}\left(\dfrac{m}{m+2}\right)^m \dfrac{f(x_n)}{f'(y_n)}\right)\left[G\left(\dfrac{f'(y_n)}{f'(x_n)}\right) + H\left(\dfrac{f(x_n)}{f'(y_n)}\right)\right], \end{cases} \tag{1.5}$$

where $G(\cdot)$ and $H(\cdot)$ are two real valued weight functions.

On the other hand, Soleymani and Babajee [7] in 2013, developed following fourth-order optimal family of methods

$$\begin{cases} y_n = x_n - \dfrac{2m}{m+2}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{4m\left(\frac{m}{m+2}\right)^m f(x_n)}{\left(\frac{m}{m+2}\right)^m (m^2+2m-4) f'(x_n) - m^2 f'(y_n)} H\left(\dfrac{f'(y_n)}{f'(x_n)}\right), \end{cases} \tag{1.6}$$

where $H(\cdot)$ is a real valued weight function.

Zhou et al. [8] in 2013, constructed another family of fourth-order methods, requiring two-function and one-derivative evaluation per iteration as follows:

$$\begin{cases} y_n = x_n - m\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - mG(v)\dfrac{f(x_n)}{f'(x_n)}, \end{cases} \tag{1.7}$$