Applied Mathematics and Computation 265 (2015) 570–573

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

New characterizations for the solution set to interval linear systems of equations

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ARTICLE INFO

MSC:
65G30
65F99

Keywords:
Interval linear equations
Solution set
Characterization

ABSTRACT

New characterizations of the points from the solution set to interval linear systems of equations are proposed, alternatives to the well-known result by Oettli and Prager. We also introduce recognizing functionals of the solution sets that determine, at a given point, aggregated quantitative measures of compatibility (consistency) between the interval data of the system.

1. Introduction

We consider interval linear systems of equations of the form

$$\begin{cases} \mathbf{a}_{11}x_1 + \mathbf{a}_{12}x_2 + \ldots + \mathbf{a}_{1n}x_n = \mathbf{b}_1, \\ \mathbf{a}_{21}x_1 + \mathbf{a}_{22}x_2 + \ldots + \mathbf{a}_{2n}x_n = \mathbf{b}_2, \\ \quad \vdots \qquad\qquad \ddots \qquad\qquad \vdots \\ \mathbf{a}_{m1}x_1 + \mathbf{a}_{m2}x_2 + \ldots + \mathbf{a}_{mn}x_n = \mathbf{b}_m, \end{cases} \tag{1}$$

or, briefly,

$$\mathbf{A}x = \mathbf{b} \tag{2}$$

with an interval $m \times n$-matrix $\mathbf{A} = (\mathbf{a}_{ij})$ and an interval m-vector $\mathbf{b} = (\mathbf{b}_i)$.

Below, we use the notation proposed in the informal standard [1]. In particular, intervals and interval values are denoted by bold letters, whereas noninterval (point) values are not specified in any special manner. Underlining $\underline{\mathbf{a}}$ and overlining $\overline{\mathbf{a}}$ denote the lower and upper endpoints of the interval $\mathbf{a}$, so that we have $\mathbf{a} = [\underline{\mathbf{a}}, \overline{\mathbf{a}}] = \{ a \in \mathbb{R} \mid \underline{\mathbf{a}} \leqslant a \leqslant \overline{\mathbf{a}} \}$. Additionally,

$$\text{mid}\,\mathbf{a} = \tfrac{1}{2}(\overline{\mathbf{a}} + \underline{\mathbf{a}}) \quad \text{is the midpoint of the interval,}$$

$$\text{rad}\,\mathbf{a} = \tfrac{1}{2}(\overline{\mathbf{a}} - \underline{\mathbf{a}}) \quad \text{is the radius of the interval,}$$

$$\langle \mathbf{a} \rangle = \begin{cases} \min\{\,|\overline{\mathbf{a}}|, |\underline{\mathbf{a}}|\,\}, & \text{if } 0 \notin \mathbf{a}, \\ 0, & \text{otherwise,} \end{cases} \quad \begin{array}{l} \text{is the mignitude of the interval,} \\ \text{i.e., the smallest distance from its points to zero.} \end{array}$$

The mignitude is, in a sense, an antipode of the absolute value (magnitude) of the interval, defined as $|\mathbf{a}| = \max_{a \in \mathbf{a}} |a| = \max\{\,|\underline{\mathbf{a}}|, |\overline{\mathbf{a}}|\,\}$.

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http://dx.doi.org/10.1016/j.amc.2015.05.029
0096-3003/

Interval linear system of equations of the form (1) and (2),

$$\boldsymbol{A}x = \boldsymbol{b},$$

is a family of point (non-interval) linear systems $Ax = b$ of the same structure with $A = (a_{ij}) \in \boldsymbol{A}$ and $b = (b_i) \in \boldsymbol{b}$, i. e., such that $a_{ij} \in \boldsymbol{a}_{ij}$ and $b_i \in \boldsymbol{b}_i$ for all the indices i, j. The *solution set* to the interval linear system is defined as the set

$$\Xi(\boldsymbol{A}, \boldsymbol{b}) = \{ x \in \mathbb{R}^n \mid (\exists A \in \boldsymbol{A})(\exists b \in \boldsymbol{b})(Ax = b) \},$$

that is, the set of all solutions to every point linear system $Ax = b$ whose coefficients and right-hand-sides belong to $\boldsymbol{A}$ and $\boldsymbol{b}$ respectively. The set $\Xi(\boldsymbol{A}, \boldsymbol{b})$ is often referred to as *the united solution set*, since there exist the other solution sets to the systems (1) and (2) (see [2]). We do not consider them in our text, thus using the brief term "solution set".

An analytic description of the solution set to interval linear systems is given by the result obtained by Oettli and Prager in 1964 [3]:

$$x \in \Xi(\boldsymbol{A}, \boldsymbol{b}) \quad \Leftrightarrow \quad \left| (\mathrm{mid}\, \boldsymbol{A})\, x - \mathrm{mid}\, \boldsymbol{b} \right| \le (\mathrm{rad}\, \boldsymbol{A})\, |x| + \mathrm{rad}\, \boldsymbol{b}, \tag{3}$$

where the operations mid, rad and $|\cdot|$ are applied to interval vectors and matrices in componentwise and elementwise manner, while the vector inequality is understood as the same inequality between every component. The purpose of our paper is to present new characterizations of the solution set $\Xi(\boldsymbol{A}, \boldsymbol{b})$, different from the result by Oettli–Prager.

2. New characterizations

Our starting point is the following assertion, which is due to Beeck [4]:

The Beeck characterization. *A point $x \in \mathbb{R}^n$ belongs to the solution set $\Xi(\boldsymbol{A}, \boldsymbol{b})$ if and only if $\boldsymbol{A} \cdot x \cap \boldsymbol{b} \neq \varnothing$, i. e., the interval vectors $\boldsymbol{A} \cdot x$ and $\boldsymbol{b}$ have nonempty intersection.*

In the formulation of the Beeck characterization, the product $\boldsymbol{A} \cdot x$ is understood as an interval-arithmetic product. Henceforth, we will omit the multiplication sign "$\cdot$", writing just $\boldsymbol{A}x$ instead of $\boldsymbol{A} \cdot x$. Specifically, the ith component of the vector $\boldsymbol{A}x$ is, by definition, $\sum_{j=1}^{n} \boldsymbol{a}_{ij} x_j$, where all the operations are those from the classical interval arithmetic (see e. g., [5,6]).

Testing the Beeck characterization for a point $\tilde{x} \in \mathbb{R}^n$ amounts to examination whether the interval boxes $\boldsymbol{A}\tilde{x}$ and $\boldsymbol{b}$ intersect with each other in the space $\mathbb{R}^n$ (see Fig. 1). To render the geometric ideas of the Beeck characterization into analytical language, let us consider the one-dimensional case first, i. e., intersection of two one-dimensional real intervals $\boldsymbol{p}$ and $\boldsymbol{q}$.

Let us shift the whole situation by $\mathrm{mid}\,\boldsymbol{q}$, that is, let us consider the intervals $(\boldsymbol{p} - \mathrm{mid}\,\boldsymbol{q})$ and $(\boldsymbol{q} - \mathrm{mid}\,\boldsymbol{q})$ instead of the original $\boldsymbol{p}$ and $\boldsymbol{q}$. It is obvious that the intervals $(\boldsymbol{p} - \mathrm{mid}\,\boldsymbol{q})$ and $(\boldsymbol{q} - \mathrm{mid}\,\boldsymbol{q})$ intersect each other if and only if this is true for the intervals $\boldsymbol{p}$ and $\boldsymbol{q}$.

The intervals $(\boldsymbol{p} - \mathrm{mid}\,\boldsymbol{q})$ and $(\boldsymbol{q} - \mathrm{mid}\,\boldsymbol{q})$ have nonempty intersection if and only if the mignitude of the first interval does not exceed the absolute value of the second one. Since the absolute value of $(\boldsymbol{q} - \mathrm{mid}\,\boldsymbol{q})$ is evidently $\mathrm{rad}\,\boldsymbol{q}$, then the above means that

$$\boldsymbol{p} \cap \boldsymbol{q} \neq \varnothing \quad \Leftrightarrow \quad \langle\, \boldsymbol{p} - \mathrm{mid}\,\boldsymbol{q} \,\rangle \le \mathrm{rad}\,\boldsymbol{q} \tag{4}$$

(see Fig. 2). Also, the last inequality can be rewritten as

$$\mathrm{rad}\,\boldsymbol{q} - \langle\, \boldsymbol{p} - \mathrm{mid}\,\boldsymbol{q} \,\rangle \ \ge\ 0. \tag{5}$$

Insofar as the multidimensional interval boxes are direct products of one-dimensional intervals, then, testing the Beeck characterization and examining intersection of $\boldsymbol{A}\tilde{x}$ and $\boldsymbol{b}$, we can claim that

$$\boldsymbol{A}\tilde{x} \cap \boldsymbol{b} \neq \varnothing \quad \Leftrightarrow \quad \langle (\boldsymbol{A}\tilde{x})_i - \mathrm{mid}\,\boldsymbol{b}_i \rangle \le \mathrm{rad}\,\boldsymbol{b}_i, \qquad i = 1, 2, \dots, m. \tag{6}$$

If we fix the understanding that the operations $\langle \cdot \rangle$, $\mathrm{rad}(\cdot)$ and the inequality "$\le$" are applied to interval vectors in a componentwise manner, then the system (6) can be reduced to a concise form:

$$\boldsymbol{A}\tilde{x} \cap \boldsymbol{b} \neq \varnothing \quad \Leftrightarrow \quad \left\langle \boldsymbol{A}\tilde{x} - \mathrm{mid}\,\boldsymbol{b} \right\rangle \le \mathrm{rad}\,\boldsymbol{b}.$$

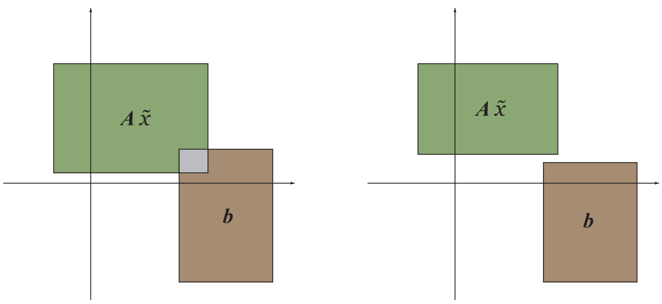

Fig. 1. Mutual disposition of the boxes $\boldsymbol{A}\tilde{x}$ and $\boldsymbol{b}$.