Applied Mathematics and Computation 265 (2015) 574–601

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Convergence analysis of discrete legendre spectral projection methods for hammerstein integral equations of mixed type

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ARTICLE INFO

Keywords:
Hammerstein integral equations
Legendre polynomials
Discrete Galerkin
Discrete collocation
Numerical quadrature
Superconvergence rates

ABSTRACT

In this paper, we consider the discrete Legendre spectral Galerkin and discrete Legendre spectral collocation methods to approximate the solution of mixed type Hammerstein integral equation with smooth kernels. The convergence of the discrete approximate solutions to the exact solution is discussed and the rates of convergence are obtained. We have shown that, when the quadrature rule is of certain degree of precision, the rates of convergence for the Legendre spectral Galerkin and Legendre spectral collocation methods are preserved. We obtain superconvergence rates for the iterated discrete Legendre Galerkin solution. By choosing the collocation nodes and quadrature points to be same, we also obtain superconvergence rates for the iterated discrete Legendre collocation solution.

1. Introduction

In this section, we consider the following Hammerstein integral equation of mixed type

$$x(t) - \sum_{i=1}^{m} \int_{-1}^{1} k_i(t,s)\psi_i(s,x(s))ds = f(t), \quad -1 \le t \le 1, \tag{1.1}$$

where f, k_i and ψ_i $(1 \le i \le m)$ are known functions and x is the unknown solution to be found in a Banach space $\mathbb{X}$.

Mixed type Hammerstein integral Eq. (1.1) often arises as a reformulation of boundary value problems with certain nonlinear boundary conditions. Several numerical methods, such as boundary element methods, various spectral methods are available in literature to solve nonlinear integral equations (see [2,3,4,6,7,14,23]). The existence and uniqueness of the solution and convergence analysis of various spectral approximations of nonlinear systems of integral equations are conveyed in [11,22,26]. The Galerkin, collocation and their discretized versions are the most commonly used projection methods for finding numerical solutions of the integral equation of type (1.1) (see [8–12]). Superconvergence results of various projection methods for solving nonlinear Fredholm integral equations can be found in ([8,9,17,18,20]).

Projection methods for solving the equation of type (1.1) using piecewise polynomials were studied by Ganesh and Joshi [11] and a discretized version of those methods was introduced in [12]. Polynomially-based projection methods for the integral equation of type (1.1) were discussed in [9].

The projection methods for (1.1) lead to algebraic non-linear system, in which the coefficients are integrals, appeared due to inner products and the integral operator. These integrals are almost always evaluated numerically. However, in all the projection

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http://dx.doi.org/10.1016/j.amc.2015.05.100

methods for Hammerstein integral equations and mixed type Hammestein integral Eq. (1.1), it is assumed that the integrals appearing in the approximation schemes are evaluated exactly (see [2,9,11,19]). Thus, in all these methods the effect of error due to the numerical integration has been ignored. This motivates to solve these nonlinear systems after replacing the integrals by appropriate numerical integration formula. Replacement of these integrals by numerical quadrature rule gives rise to the discrete projection methods. The effect of quadrature error on the convergence rates of the approximate solutions is now considered in these discrete projection methods (see [3,4,12]). Now to apply discrete projection methods for solving the Eq. (1.1), one can use either piecewise polynomials or global polynomials as basis functions of the approximating subspaces. In [12], Ganesh and Joshi obtained convergence rates for the approximate solution of piecewise polynomial based discrete collocation method, when the quadrature rule is of certain degree of precision. However in case of piecewise polynomial based projection methods, to get better accuracy in approximate solutions, one has to increase the number of partition points. This leads to solve a large system of nonlinear equations, which is computationally very expensive. Use of global polynomials imply smaller nonlinear systems, something which is highly desirable in practical computations. Hence, to overcome the computational complexities encountered in the existing piecewise polynomial based discrete projection methods, we apply polynomially-based discrete projection methods to solve Fredholm–Hammerstein integral equation of mixed type (1.1).

In this paper, we apply discrete Galerkin and discrete collocation methods to solve mixed type Fredholm–Hammerstein integral Eq. (1.1) using global polynomial basis functions. In particular here, we choose to use Legendre polynomials, which can be generated recursively with ease and possess nice property of orthogonality. Further, these Legendre polynomials are less expensive computationally compared to piecewise polynomial basis functions. By choosing a sufficiently accurate numerical quadrature rule, we show that the discrete Legendre Galerkin and discrete Legendre collocation solutions of the Eq. (1.1) converge with the optimal order $\mathcal{O}(n^{-r})$ in both infinity and L^2-norm, and the iterated discrete Legendre Galerkin solution converges with the order $\mathcal{O}(n^{-2r})$ in both infinity and L^2-norm, whereas the iterated discrete Legendre collocation solution converges with the order $\mathcal{O}(n^{-r})$ in both infinity and L^2-norm, n being the highest degree of the polynomial approximation and r being the smoothness of the kernels, the nonlinear functions ψ_i, the right hand side function f and the exact solution. Thus, we obtain superconvergence rates for the iterated discrete Legendre Galerkin method. In a particular case, when the quadrature points and collocation nodes are chosen to be same, the iterated discrete Legendre collocation solution converges with the order $\mathcal{O}(n^{-2r})$ in both infinity and L^2-norm. Thus in such case we also obtain superconvergence rates for the iterated discrete Legendre collocation method.

We organize this paper as follows. In Section 2, we set up notations and discuss the discrete Legendre Galerkin and discrete Legendre collocation methods for mixed type Hammerstein integral equations with smooth kernels. In Section 3, we discuss the existence of the approximate and iterated approximate solutions and their convergence rates. In Section 4, we illustrate our results by numerical examples. Throughout this paper, we assume that c is a generic constant.

2. Discrete Legendre spectral Galerkin and discrete Legendre spectral collocation methods: Hammerstein integral equations of mixed type with smooth kernel

In this section, we describe the discrete Galerkin and discrete collocation methods for solving Hammerstein integral equation of mixed type using Legendre polynomial basis functions.

Let $\mathbb{X} = \mathcal{C}[-1,1]$. For $u \in \mathbb{X}$, we define

$$\|u\|_\infty = \sup_{t\in[-1,1]} |u(t)| \text{ and } \|u\|_{L^2} = \left(\int_{-1}^{1} |u(t)|^2 dt\right)^{\frac{1}{2}}.$$

Consider the following Hammerstein integral equation of mixed type

$$x(t) - \sum_{i=1}^{m} \int_{-1}^{1} k_i(t,s)\psi_i(s,x(s))ds = f(t), \quad -1 \le t \le 1, \tag{2.1}$$

where k_i, f and ψ_i are known functions and x is the unknown function to be determined. Let

$$(\mathcal{K}_i x)(t) = \int_{-1}^{1} k_i(t,s)x(s)\,ds, \quad x \in \mathbb{X}. \tag{2.2}$$

For our convenience, we consider a nonlinear operator $\Psi_i : \mathbb{X} \to \mathbb{X}$ defined by

$$(\Psi_i x)(t) := \psi_i(t, x(t)), \quad t \in [-1,1], \quad x \in \mathbb{X}. \tag{2.3}$$

Then the Eq. (2.1) can be written as

$$x - \sum_{i=1}^{m} \mathcal{K}_i \Psi_i(x) = f. \tag{2.4}$$

Throughout the paper, the following assumptions are made on f, $k_i(.,.)$ and $\psi_i(., x(.))$:

(i) $f \in \mathcal{C}[-1,1]$.
(ii) $\lim_{t\to t'} \|k_i(t,.) - k_i(t',.)\|_\infty = 0, \ t, t' \in [-1,1], \ 1 \le i \le m$.
(iii) For $1 \le i \le m$, $k_i(.,.) \in \mathcal{C}([-1,1]\times[-1,1])$. Let $m_i = \sup_{t,s\in[-1,1]} |k_i(t,s)| < \infty$ and $M_1 = \max_{1\le i\le m} m_i$.