Applied Mathematics and Computation 265 (2015) 619–629

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Inertia of complex unit gain graphs

Guihai Yu[a,*], Hui Qu[a], Jianhua Tu[b]

[a] Department of Mathematics, Shandong Institute of Business and Technology, Yantai, Shandong 264005, China
[b] School of Science, Beijing University of Chemical Technology, Beijing 100029, China

ARTICLE INFO

Keywords:
Inertia
Complex unit gain graph
Tree
Unicyclic graph

ABSTRACT

Let $\mathbb{T} = \{z \in C : |z| = 1\}$ be a subgroup of the multiplicative group of all nonzero complex numbers $\mathbb{C}^{\times}$. A $\mathbb{T}$-gain graph is a triple $\Phi = (G, \mathbb{T}, \varphi)$ consisting of a graph $G = (V, E)$, the circle group $\mathbb{T}$ and a gain function $\varphi : \overrightarrow{E} \to \mathbb{T}$ such that $\varphi(e_{ij}) = \varphi(e_{ji})^{-1} = \overline{\varphi(e_{ji})}$. The adjacency matrix $A(\Phi)$ of the $\mathbb{T}$-gain graph $\Phi = (G, \varphi)$ of order n is an $n \times n$ complex matrix (a_{ij}), where

$$a_{ij} = \begin{cases} \varphi(e_{ij}), & \text{if } v_i \text{ is adjacent to } v_j, \\ 0, & \text{otherwise.} \end{cases}$$

Evidently this matrix is Hermitian. The *inertia* of Φ is defined to be the triple $In(\Phi) = (i_+(\Phi), i_-(\Phi), i_0(\Phi))$, where $i_+(\Phi), i_-(\Phi), i_0(\Phi)$ are numbers of the positive, negative and zero eigenvalues of $A(\Phi)$ including multiplicities, respectively. In this paper we investigate some properties of inertia of $\mathbb{T}$-gain graph.

1. Introduction

The gain graph is a graph whose edges are labeled orientably by elements of a group S. This means that, if an edge e in one direction has label s (a group element in S), then in the other direction it has label s^{-1} (the invertible element of s in S). The group S is called the *gain group*. A gain graph is a generalization of a signed graph where the gain group S has only two elements 1 and -1. This can be referred to [35]. In fact, a gain graph is a weighted digraph where all weights on arcs are elements in a group S. In this paper we shall consider a special gain graph — complex unit gain graph.

Throughout this paper we only consider simple graphs, i.e., without multiedges and loops. A graph G is denoted by $G = (V, E)$, where V is the vertex set and E is the edge set. Let $\overrightarrow{E}$ be the set of oriented edges. It is evident that this set contains two copies of each edge with opposite orientations. For convenience, we write $e_{v_i v_j}$ (or for short e_{ij}) for the oriented edge from v_i to v_j. Let $\mathbb{T} = \{z \in C : |z| = 1\}$, which is a circle group, is a subgroup of the multiplicative group of all nonzero complex numbers $\mathbb{C}^{\times}$. A $\mathbb{T}$-*gain graph (complex unit gain graph)* is a graph with additional structure that each orientation of an edge is given a complex unit, called a *gain*, which is the inverse of the complex unit assigned to the opposite orientation. Here $\mathbb{T}$ is the gain group. Formally, a $\mathbb{T}$-gain graph is a triple $\Phi = (G, \mathbb{T}, \varphi)$ consisting of a graph $G = (V, E)$, the gain group $\mathbb{T}$ and a gain function $\varphi : \overrightarrow{E} \to \mathbb{T}$ such that $\varphi(e_{ij}) = \varphi(e_{ji})^{-1} = \overline{\varphi(e_{ji})}$. G is called the *underlying graph* of the $\mathbb{T}$-gain graph Φ. For brevity, we write $\Phi = (G, \varphi)$ for a $\mathbb{T}$-gain graph. The adjacency matrix $A(\Phi)$ of the $\mathbb{T}$-gain graph $\Phi = (G, \varphi)$ of order n is an $n \times n$ complex matrix (a_{ij}), where

$$a_{ij} = \begin{cases} \varphi(e_{ij}), & \text{if } v_i \text{ is adjacent to } v_j, \\ 0, & \text{otherwise.} \end{cases}$$

* Corresponding author. Tel.: +86 5356903824.
E-mail addresses: yuguihai@126.com (G. Yu), quhui781111@126.com (H. Qu), tujh81@163.com (J. Tu).

http://dx.doi.org/10.1016/j.amc.2015.05.105
0096-3003/

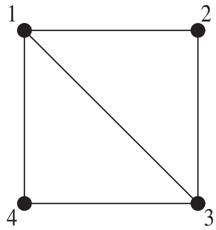

Fig. 1. An example for complex unit gain graph.

Obviously, $A(\Phi)$ is Hermitian and its eigenvalues are real.

Example 1. Given an underlying graph with four vertices {1, 2, 3, 4} (as shown in Figure 1). We shall construct a complex unit gain graph such that the gain function satisfies the following: $\varphi(e_{12}) = i = \varphi(e_{21})^{-1}$; $\varphi(e_{13}) = 1 = \varphi(e_{31})$; $\varphi(e_{14}) = \frac{1}{2} - \frac{\sqrt{3}}{2}i = \varphi(e_{41})^{-1}$; $\varphi(e_{23}) = \frac{1}{2} + \frac{\sqrt{3}}{2}i = \varphi(e_{32})^{-1}$; $\varphi(e_{34}) = \frac{\sqrt{3}}{2} + \frac{1}{2}i = \varphi(e_{43})^{-1}$, where $i = \sqrt{-1}$. The adjacency matrix is

$$\begin{pmatrix} 0 & i & 1 & \frac{1}{2} - \frac{\sqrt{3}}{2}i \\ -i & 0 & \frac{1}{2} + \frac{\sqrt{3}}{2}i & 0 \\ 1 & \frac{1}{2} - \frac{\sqrt{3}}{2}i & 0 & \frac{\sqrt{3}}{2} + \frac{1}{2}i \\ \frac{1}{2} + \frac{\sqrt{3}}{2}i & 0 & \frac{\sqrt{3}}{2} - \frac{1}{2}i & 0 \end{pmatrix}.$$

If the gain of every edge is 1, the adjacency matrix $A(\Phi)$ is exactly the adjacency matrix $A(G)$ of the underlying graph. Obviously, a simple graph is always assumed as a $\mathbb{T}$-gain graph with all positive gain 1's.

The *inertia* of Φ is defined to be the triple $In(\Phi) = (i_+(\Phi), i_-(\Phi), i_0(\Phi))$, where $i_+(\Phi), i_-(\Phi), i_0(\Phi)$ are numbers of the positive, negative and zero eigenvalues of $A(\Phi)$ including multiplicities, respectively. We call $i_+(\Phi)$ and $i_-(\Phi)$ the *positive, negative indices of inertia (abbreviated positive, negative indices)* of Φ, respectively. The number $i_0(\Phi)$ is called the *nullity* of Φ. The *rank* of an n-vertex $\mathbb{T}$-gain graph Φ, denoted by $r(\Phi)$, is defined as the rank of $A(\Phi)$. Obviously, $r(\Phi) = i_+(\Phi) + i_-(\Phi) = n - i_0(\Phi)$. Recently the inertia of $\mathbb{T}$-gain graph with all positive gain 1's has attracted some attention, for example [7,11,12,25,34]. Apart from the inertia, there are also many other spectral-based graph invariants, such as graph energies [2,6,16,17,23,24] and HOMO–LUMO index [21,27]. Some other topological molecular descriptors are also studied extensively, including distance-based [32] and degree-based [30] indices, for example, various of Wiener indices [18,26], Randić index [15,22], Kirchhoff index[9,20,28], connective eccentricity index [33], ABC index [1,14,29] and graph entropies [3,4,8,19].

The gain of a walk $W = e_{12}e_{23}\ldots e_{(l-1)l}$ is $\varphi(W) = \varphi(e_{12})\varphi(e_{23})\ldots\varphi(e_{(l-1)l})$. A switching function is a function $\zeta : V \to \mathbb{T}$. Switching the $\mathbb{T}$-gain graph $\Phi = (G, \varphi)$ by ζ means forming a new $\mathbb{T}$-gain graph $\Phi^{\zeta} = (G, \varphi^{\zeta})$ whose underlying graph is the same as G, but whose gain function is defined on an edge $e = v_iv_j$ by $\varphi^{\zeta}(e_{ij}) = \zeta(v_i)^{-1}\varphi(e_{ij})\zeta(v_j)$. Let $\Phi_1 = (G, \varphi_1)$ and $\Phi_2 = (G, \varphi_2)$ be two $\mathbb{T}$-gain graphs with the same underlying graph. We say Φ_1 and Φ are *switching equivalent*, written $\Phi_1 \sim \Phi_2$, if there exists a switching function ζ such that $\Phi_2 = \Phi_1^{\zeta}$. Switching equivalence forms an equivalence relation on gain functions for a fixed underlying graph.

An *induced subgraph* of Φ is an induced subgraph of Φ and each edge preserves the original gain in Φ. For an induced subgraph H of Φ, let $\Phi - H$ be the subgraph obtained from Φ by deleting all vertices of H and all incident edges. For $V' \subseteq V(\Phi)$, $\Phi - V'$ is the subgraph obtained from Φ by deleting all vertices in V' and all incident edges. A vertex of a graph Φ is called *pendant* if it is only adjacent to one vertex, and is called *quasi-pendant* if it is adjacent to a pendant vertex. A set M of edges in Φ is a *matching* if every vertex of Φ is incident with at most one edge in M. It is *perfect matching* if every vertex of Φ is incident with exactly one edge in M. We denote by $m_{\Phi}(i)$ the number of matchings of Φ with i edges and by $\beta(\Phi)$ the *matching number* of Φ (i.e. the number of edges of a maximum matching in Φ). For a $\mathbb{T}$-gain graph Φ on at least two vertices, a vertex $v \in V(\Phi)$ is called *unsaturated* in Φ if there exists a maximum matching M of Φ in which no edge is incident with v; otherwise, v is called *saturated* in Φ. Denote by P_n, S_n, C_n, K_n a path, a star, a cycle and a complete graph all of which are simple graphs of order n, respectively.

Nathan Reff [31] defined the adjacency, incidence and Laplacian matrices of complex unit gain graph and investigated each of them. Some eigenvalue bounds for the adjacency and Laplacian matrices were present. As a continuation of this work, in this paper we shall study some properties of inertia of adjacency matrix of $\mathbb{T}$-gain graph.

2. Three graph transformations

Definition 1. Let M be an Hermitian matrix. The three types of elementary congruence matrix operations (ECMOs) of M are defined as follows: