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Further results on digraphs with completely real Laplacian spectra

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ABSTRACT

Let Θ_n be the digraph with vertex set $V_n = \{1, 2, \ldots, n\}$ and arc set $E = \cup_{l<k}\{(k, l)|k = 2, \ldots, n\} \cup \{(k, k+1)|k = 1, \ldots, n-1\}$, and $\Theta_n^{i,j}$ the digraph obtained by deleting one arc (i, j) from Θ_n where $1 \le j < i \le n(\ n \ge 2)$. We have solved the Laplacian spectra of digraphs $\Theta_n^{i,j}$ for $i - j = 1$ and $i - j = 2$. In this paper, we further obtain the Laplacian spectra of $\Theta_n^{i,j}$ for the case $i - j = 3$. Meanwhile, $\Theta_n^{i,i-4}$ is found to have completely real spectra as well.

1. Introduction

The Laplacian matrix of a digraph Γ with vertex set $V(\Gamma) = \{1, \ldots, n\}$ and arc set $E(\Gamma)$ is the matrix $L = (l_{ij}) \in \mathbb{R}^{n\times n}$ in which, for $j \ne i$, $l_{ij} = -1$ whenever $(i, j) \in E(\Gamma)$, otherwise $l_{ij} = 0$; $l_{ii} = -\sum_{j\ne i} l_{ij}$, $i, j \in V(\Gamma)$. There have been lots of papers on Laplacian matrices of digraphs like [1–5,7–10,12,17].

Because the Laplacian matrices of digraphs are not necessarily symmetric, as distinct from those of the undirected graphs, the Laplacian eigenvalues of digraphs need not be all real. The problem of characterizing all digraphs that have completely real Laplacian spectra is difficult and yet unsolved. In general, it is not an easy problem to distinguish, in terms of graph topology, digraphs with real Laplacian spectra from those having some non-real Laplacian eigenvalues. It is easy to see that the digraphs of the latter type are guaranteed to have cycles and are called *essentially cyclic*, otherwise, the Laplacian of them will have a triangular form.

The problem of distinguishing digraphs with real and partially non-real spectra is important for applications. As mentioned in [1], in the decentralized control of multi-agent systems, the absence of non-real Laplacian eigenvalues of the communication digraph implies that the simplest consensus algorithms are devoid of oscillations. Agaev and Chebotarev [1] have pointed out that a rational approach to attacking this difficult problem is to study various classes of cyclic digraphs. In [1] a necessary and sufficient condition of essential cyclicity for the digraphs with ring structure was obtained.

Let Θ_n be the digraph with vertex set $V_n = \{1, 2, \ldots, n\}$ and arc set $E = \cup_{l<k}\{(k, l)|k = 2, \ldots, n\} \cup \{(k, k+1)|k = 1, \ldots, n-1\}$, as given in Fig. 1. Denote by L_n the Laplacian matrix of Θ_n. Note that L_n is a Hessenberg matrix which has its following form:

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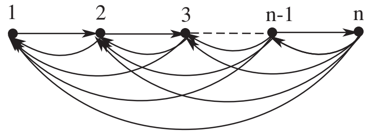

Fig. 1. Θ_n.

$$L_n = \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ -1 & -1 & 3 & -1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ -1 & -1 & -1 & \ddots & n-1 & -1 \\ -1 & -1 & -1 & \cdots & -1 & n-1 \end{pmatrix}. \tag{1.1}$$

We have shown that the digraph Θ_n is not essentially cyclic with integral Laplacian spectra $\{0, 2, 3, \ldots, n\}$. Let $\Theta_n^{i,j}$ denote the digraph $\Theta_n - (i, j)$ with $1 \le j < i \le n\,(n \ge 2)$. We have obtained the Laplacian spectra of $\Theta_n^{i,j}$ for $i - j = 1$ and $i - j = 2$. In this paper, we further solve the Laplacian spectra for the case of $i - j = 3$ and discuss the essential cyclicity of $\Theta_n^{i,j}$ when $i - j = 4$.

2. Main results

For our purpose, first we need an explicit formula for the Laplacian polynomial of $\Theta_n^{i,j}$.

Consider the following lower Hessenberg matrix $\bar{L_n}$, which is just different from L_n at the right down element,

$$\bar{L_n} = \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ -1 & -1 & 3 & -1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ -1 & -1 & -1 & \ddots & n-1 & -1 \\ -1 & -1 & -1 & \cdots & -1 & n \end{pmatrix}. \tag{2.1}$$

Denote by $Z_n(\lambda)$ the characteristic polynomial of $\bar{L_n}$. We have the following recursive relation:

$$Z_n(\lambda) = \lambda \prod_{j=2}^{n} (\lambda - j) - Z_{n-1}(\lambda). \tag{2.2}$$

Let $\varepsilon_i \in \mathbb{R}^n$ denote the row vector with ith component 1 and all other components 0. Set $E_{ij} = \varepsilon_i^T \varepsilon_j$, that is, the matrix of order n with (i, j)th entry 1 and all remaining entries 0.

Let $\mu_\Gamma(\lambda)$ denote the Laplacian characteristic polynomial of Γ, i.e. $\mu_\Gamma(\lambda) = |\lambda I - L(\Gamma)|$.

Theorem 2.1. *Let $\Theta_n^{i,j}$ denote the digraph $\Theta_n - (i, j)$ with $1 \le j < i \le n\,(n \ge 2)$. Then*

$$\mu_{\Theta_n^{i,j}}(\lambda) = \begin{cases} h(\lambda) + \lambda(\lambda - i) \prod_{l=i+2}^{n} (\lambda - l) \sum_{k=j}^{i-1} (-1)^{i+k+1} \prod_{l=2}^{k} (\lambda - l), & i < n-1; \\ h(\lambda) + \lambda(\lambda - n + 1) \sum_{k=j}^{n-2} (-1)^{n+k} \prod_{l=2}^{k} (\lambda - l), & i = n-1; \\ h(\lambda) + \lambda \sum_{k=j}^{n-1} (-1)^{n+k+1} \prod_{l=2}^{k} (\lambda - l), & i = n. \end{cases}$$

where $h(\lambda) = \lambda \prod_{l=2}^{n} (\lambda - l)$.

Proof. Let $L = L(\Theta_n^{i,j})$. First consider the case $i < n - 1$. Recall that $\Theta_n^{i,j}$ is obtained from Θ_n by deleting an arc (i, j); then $L = L(\Theta_n) + E_{ij} - E_{ii}$. Thus we have

$$\det(\lambda I_n - L) = \det(\lambda I_n - L_n) + \det(\tilde{L_n}), \tag{2.3}$$